7 Planarity

Embeddings & Euler's Formula

Planar Embedding: If G is a graph, an *embedding of G in the plane* is a function ϕ which assigns each vertex of G a distinct point in the plane and assigns to each edge e with ends u, v a simple rectifiable curve with ends $\phi(u)$ and $\phi(v)$ so that this curve minus its ends is disjoint from the image of $V(G) \cup (E(G) \setminus \{e\})$.

Planar and Plane: A *plane* graph is a graph G together with an embedding of G in the plane. A graph is *planar* if there exists an embedding of it in the plane.

Faces: If G is a plane graph, then the space obtained from the plane by removing all points in the image of G consists of finitely many connected components, each of which is called a *face* of G. We let $F(G)$ denote the set of all faces of G. Every face $a \in F(G)$ is bounded by a closed walk (not necessarily a cycle) called a *boundary walk*. The *size* of a, denoted $size(a)$, is the length of this walk, and an edge or vertex is *incident* with a if it appears in this walk. Note that every edge appears exactly twice in the boundary walks of the faces.

Theorem 7.1 (Euler's Formula) *If G is a connected plane graph, then*

$$|V(G)| - |E(G)| + |F(G)| = 2.$$

Proof: We proceed by induction on $|E(G)|$. If $|E(G)| = 0$, then $|V(G)| = 1 = |F(G)|$ so the formula holds. For the inductive step, let G be a plane graph with $|E(G)| > 0$ and choose $e \in E(G)$. If e is a non loop edge, then contracting e results in a plane graph (namely $G \cdot e$) with the same number of faces but one fewer edge and one fewer vertex. So, the result follows by applying induction to $G \cdot e$. If e is a loop edge, then e separates the plane[1], so there are two distinct faces incident with e, and deleting e results in a plane graph (namely $G - e$) with the same number of vertices but one fewer edge and one fewer face. So, the result follows by applying induction to $G - e$. $\square$

[1] The seemingly obvious statement that every simple closed curve in the plane separates it into two regions is the Jordan Curve Theorem and is surprisingly difficult to prove. We shall take this as an assumption.

Duality

Duality: Let G be a plane graph, and construct a new plane graph G^* as follows. For each face $a \in F(G)$, add a vertex a^* in a. For each edge $e \in E(G)$ which lies in the boundary walk of the faces a, b, add an edge from a^* to b^* which crosses e but is otherwise disjoint from the image of G (if e appears twice in the boundary walk of a, then e^* is a loop at a^*). This can be done so that G^* is a plane graph, and we call any plane graph constructed in this manner a *dual* of G. As suggested by the name, if G^* is a dual of G, then G is a dual of G^*. This gives correspondences $V(G) \sim F(G^*)$, $E(G) \sim E(G^*)$, and $F(G) \sim V(G^*)$, and for any set X of vertices, edges, or faces of G we let X^* denote the corresponding set of elements in G^*.

Observation 7.2 *Let G, G^* be dual plane graphs.*

(i) *If $a \in F(G)$, then $size(a) = deg(a^*)$.*

(ii) $\sum_{a \in F(G)} size(a) = 2|E(G)|$.

Proof: (i) follows immediately from the definitions. For (ii), note that

$$2|E(G)| = 2|E(G^*)| = \sum_{a^* \in V(G^*)} deg(a^*) = \sum_{a \in F(G)} size(a). \qquad \Box$$

Proposition 7.3 *If G, G^* are connected dual plane graphs, then G is bipartite if and only if G^* is Eulerian.*

Proof: By possibly adding parallel edges to G, we may assume that every edge of G appears in the boundary walks of two distinct faces. For the "only if" direction, note that if G is bipartite, then every boundary walk has even length, so every vertex of G^* has even degree. For the "if" direction, assume G^* is Eulerian, let C be a cycle in G, and let $A \subseteq F(G)$ be the set of faces which lie inside C. Now, $E(C)$ is the symmetric difference of the edge sets in the boundary walks of faces in A. Since each such boundary walk has even length, $|E(C)|$ is even as well. $\Box$

Bond: A *bond* is a nonempty edge cut which is minimal (with respect to inclusion). In a connected graph, an edge cut $\delta(X)$ is a bond if and only if setting $Y = V(G) \setminus X$, both $G - X$ and $G - Y$ are connected.

Proposition 7.4 *If $G = (V, E)$ and $G^* = (V^*, E^*)$ are dual plane graphs, then*

(i) *$C \subseteq E$ is the edge set of a cycle (in G) if and only if C^* is a bond (in G^*).*

(ii) *if $\{S, T\}$ is a partition of E, then (V, T) is a spanning tree of G if and only if (V^*, S^*) is a spanning tree of G^*.*

Proof: For (i), we begin by proving the "only if" direction. Let C be the edge set of a cycle in G. Then the image of C separates the plane into two connected components, and this gives a partition of the faces of G. Since any two faces in the same component may be joined by a curve in the plane disjoint from $V(G) \cup C$, it follows that C^* is a bond of G^*. For the "if" direction, suppose that C^* is a bond of G^*. Then C^* gives us a partition of the vertices of G^* (or equivalently a partition of the faces of G) into two connected components. The edges of G which lie on the boundary of these two components are precisely those in C, so C is the edge set of a cycle, as desired.

For (ii), we have:

(V, T) is a spanning tree of G.

$\Leftrightarrow$ T does not contain the edge set of a cycle and S does not contain a bond.

$\Leftrightarrow$ T^* does not contain a bond and S^* does not contain the edge set of a cycle.

$\Leftrightarrow$ (V^*, S^*) is a spanning tree of G^*. $\square$

Another proof of Euler's Formula: Let G be a connected plane graph with dual G^*. Choose a partition $\{S, T\}$ of $E(G)$ so that T is the edge set of a spanning tree in G. Then S^* is the edge set of a spanning tree in G^* so we have

$$2 = (|V(G)| - |T|) + (|V(G^*)| - |S|) = |V(G)| - |E(G)| + |F(G)|. \qquad \square$$

Applications of Euler's Formula

Triangulation: A *triangulation* of the plane is a plane graph in which every face has size three.

Lemma 7.5 *If G is a simple planar graph, then*

(i) $6 \leq 3|V(G)| - |E(G)|$ *with equality for triangulations.*

(ii) $4 \leq 2|V(G)| - |E(G)|$ *if G has no triangle.*

Proof: By possibly adding edges, we may assume that G is connected. For (i), note that if every face has size ≥ 3, part (ii) of Observation 7.2 implies $2|E(G)| \geq 3|F(G)|$. Plugging this into Euler's Formula gives $6 = 3|V(G)| - 3|E(G)| + 3|F(G)| \leq 3|V(G)| - |E(G)|$. This formula holds with equality if every face is a triangle, giving us (i). The proof of (ii) is similar to that of (i) except that every face has size ≥ 4 so we get $2|E(G)| \geq 4|F(G)|$ by Observation 7.2 and then Euler's Formula yields $4 = 2|V(G)| - 2|E(G)| + 2|F(G)| \leq 2|V(G)| - |E(G)|$. $\square$

Observation 7.6 *The graphs K_5 and $K_{3,3}$ are not planar.*

Proof: Since K_5 has 5 vertices and 10 edges, it cannot be planar by (i) of the previous lemma. Since $K_{3,3}$ has 6 vertices and 9 edges, it cannot be planar by (ii) of this lemma. $\square$

Theorem 7.7 *Let G be a connected d-regular plane graph and assume that G^* is k-regular. If $d, k \geq 3$, then G is one of Tetrahedron, Cube, Octahedron, Dodecahedron, Icosahedron.*

Proof: Let $v = |V(G)|$, $e = |E(G)|$, and $f = |F(G)|$. It follows from Observation 7.2 that $2e = kf$ and from the degree sum formula that $2e = dv$. Substituting into Euler's formula we get

$$2 = v - e + f = \tfrac{2e}{d} - e + \tfrac{2e}{k}$$

so by elementary manipulations we have:

$$\tfrac{1}{d} + \tfrac{1}{k} = \tfrac{1}{e} + \tfrac{1}{2}$$

If $k, d \geq 4$ then $\frac{1}{d} + \frac{1}{k} \leq \frac{1}{2}$ so the above equation cannot be satisfied. Similarly, if one of d or k is equal to 3 and the other is ≥ 6, then $\frac{1}{d} + \frac{1}{k} \leq \frac{1}{2}$ so this equation cannot be satisfied. Thus, the only possible values for (d, k) are $(3, 3)$, $(3, 4)$, $(4, 3)$, $(3, 5)$, and $(5, 3)$. Further, our equation implies that in these cases, G must (respectively) have 6, 12, 12, 30, and 30 edges. It then follows from an easy case analysis that Tetrahedron, Octahedron, Cube, Dodecahedron, and Icosahedron are the only possibilities. $\square$

The Structure of Planar Graphs

Minor: Let G be a graph. Any graph which can be formed from G by a sequence of vertex and edge deletions and edge contractions is called a *minor* of G. Note that if H is a graph of maximum degree 3, then G has an H minor if and only if G contains a subdivision of H.

Series-Parallel: A graph G is *series-parallel* if G can be constructed from the null graph by applying the following operations (repeatedly):

- adding a vertex of degree ≤ 1.
- adding a loop or parallel edge.
- subdividing an edge.

Theorem 7.8 *A graph is series-parallel if and only if it has no K_4 minor.*

Proof: The "only if" direction follows by an easy induction argument, since none of the above operations can introduce a K_4 minor. We prove the "if" direction by induction on $|V(G)| + |E(G)|$. As a base, note that this is trivial when G is null. For the inductive step, let G be a non null graph without a K_4 minor. It follows from Theorem 6.8 that G must have either a parallel edge or a vertex of degree ≤ 2. If G has a parallel edge or a vertex of degree ≤ 1, then by deleting this element and applying induction we deduce that G is series parallel. If $v \in V(G)$ has degree two, then the result follows by applying the reverse operation of subdivision to v (i.e. delete v and then add a new edge between its neighbors) and then applying induction. $\square$

Lemma 7.9 *If G is a plane graph, then G is 2-connected if and only if every face of G is bounded by a cycle.*

Proof: We prove the "only if" direction by way of the contrapositive. Assume that there is a face a which is not bounded by a cycle. Choose a vertex v so that the boundary walk of a passes through the vertex v twice. Then we may draw a closed curve starting and ending at v with interior contained in a. This curve separates the plane into two components each of which must contain a vertex of G, so we find that v is a cut vertex. Thus, G is not 2-connected.

For the "if" direction we also prove the contrapositive. Let G be a plane graph which is not 2-connected. Choose a proper 1-separation (H_1, H_2) of G with $V(H_1) \cap V(H_2) = \{v\}$. Now, H_1 and H_2 are plane graphs meeting only at the vertex v with H_2 embedded in a face of H_1. It follows that there is a face a of G with boundary walk passing through v twice, so a is not bounded by a cycle. $\square$

Lemma 7.10 *Let C be a cycle and let $X, Y \subseteq V(C)$. Then one of the following holds:*

(i) $|X| \leq 1$ *or* $|Y| \leq 1$.

(ii) $X = Y$.

(iii) *There exist $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ so that x_1, y_1, x_2, y_2 are distinct and occur on C in this order.*

(iv) *There are vertices $u, v \in V(C)$ so that if P, Q are the two paths of C between u and v, then $X \subseteq V(P)$ and $Y \subseteq V(Q)$.*

Proof: We shall assume that $|X|, |Y| \geq 2$ and $X \neq Y$ as otherwise one of (i) or (ii) holds. By possibly switching X and Y, we may assume that $X \setminus Y \neq \emptyset$ and choose $x_1 \in X \setminus Y$. Let y_1 be the first vertex in Y clockwise from x_1 and let y_2 be the first vertex in Y counterclockwise from x_1. Since $|Y| \geq 2$ we have that $y_1 \neq y_2$. Let P, Q be the two paths of C between y_1 and y_2 and assume that x_1 lies on P. If $X \subseteq V(P)$, then (iv) holds, otherwise (iii) holds. $\square$

Theorem 7.11 (Kuratowski-Wagner) *A graph is planar if and only if it has no $K_{3,3}$ or K_5 minor.*

Proof: We first prove the "only if" direction. If we take a plane graph and either delete an edge or vertex, or contract an edge, the resulting graph may still be embedded in the plane. Thus, every minor of a planar graph is planar. Since K_5 and $K_{3,3}$ are not planar (Observation 7.6), it follows that no graph with a K_5 or $K_{3,3}$ minor is planar.

For the "if" direction, we let G be a graph with no K_5 or $K_{3,3}$ minor and we proceed by induction on $|V(G)| + |E(G)|$. If G is not connected, then by applying the induction hypothesis to each component, we obtain a plane embedding of each component, and by combining these, we get a plane embedding of G. Thus, we may assume G is connected. We will proceed with a sequence of similar (but more complicated) steps.

Suppose that G is not 2-connected.

In this case, we may choose a nontrivial 1-separation (H_1, H_2) of G (unless $|V(G)| \leq 2$ in which case the theorem is trivial). Since H_1 and H_2 have no K_5 or $K_{3,3}$ minor, by induction, we may embed them in the plane. Combining these embeddings gives a plane embedding of G. Thus, we may assume G is 2-connected.

Suppose that G is not 3-connected.

In this case, choose a nontrivial 2-separation (H_1, H_2) of G with $\{u, v\} = V(H_1) \cap V(H_2)$ (unless $|V(G)| \leq 3$ in which case the result is trivial). Add a new edge uv to H_1 and H_2 to form H_1^+ and H_2^+. Choose vertices $z_1 \in V(H_1) \setminus V(H_2)$ and $z_2 \in V(H_2) \setminus V(H_1)$ and apply Menger's theorem to choose two internally disjoint paths from z_1 to z_2. It follows from the existence of these paths that H_1^+ and H_2^+ are minors of G (for instance, to see that H_1^+ is a minor, delete all vertices and edges of H_2 not on the two paths chosen above and then contract all but one of the edges in H_2 which remain). It follows from this that H_1^+ and H_2^+ have no K_5 or $K_{3,3}$ minor, so by induction, we may choose planar embeddings of them. By combining these embeddings on the edge uv and then removing it, we obtain a plane embedding of G. Thus, we may assume that G is 3-connected.

Suppose that G has an edge uv so that $G - \{u, v\}$ is not 2-connected.

If such an edge exists, then we may choose a nontrivial 3-separation (H_1, H_2) of $G - uv$ where $V(H_1) \cap V(H_2) = \{u, v, w\}$. For $i = 1, 2$ let H_i^+ be the graph obtained from H_i by adding a new vertex adjacent to u, v, and w. Then choose vertices $z_1 \in V(H_1) \setminus V(H_2)$ and $z_2 \in V(H_2) \setminus V(H_1)$ and apply Menger's theorem to choose three internally disjoint paths from z_1 to z_2. It follows from the existence of these paths that H_1^+ and H_2^+ are minors of G (for instance, to see that H_1^+ is a minor, delete all vertices and edges of H_2 not on the three paths chosen, and then contract all edges in H_2 except for one on each of the paths). It follows that H_1^+ and H_2^+ have no K_5 or $K_{3,3}$ minor, so by induction, we may choose plane embeddings of them. By combining these embeddings, we may obtain a plane embedding of G. Thus, we may assume that G has no such edge uv.

We now have sufficient connectivity for our inductive procedure. Choose an edge xy of G, let $G' = G \cdot xy$, let z be the vertex formed by contracting xy, and let $G'' = G - \{x, y\} = G' - z$. Now, G' has no K_5 or $K_{3,3}$ minor, so by induction, we may choose a planar embedding of it. Furthermore, it follows from our assumptions that $G'' = G - \{x, y\}$ is 2-connected, so (by Lemma 7.9) the face of G'' which contains the vertex z is bounded by a cycle C. Thus, all neighbors of x and y in G lie on the cycle C. We now start with our embedding of $G'' = G - \{x, y\}$ and try to extend this to a embedding of G. Let X be the set of neighbors of x in G and Y be the set of neighbors of y (so X and Y are subsets of $V(C)$). We now apply Lemma 7.10 to C for X and Y. If either (i) holds or (ii) holds and $|X| = |Y| = 2$, then we may extend our embedding of G'' to a plane embedding of G. If (ii) holds with $|X| = |Y| \geq 3$, then G contains a K_5 minor, contradicting our assumption. If (iii) holds, then G contains a $K_{3,3}$ minor, contradicting our assumption. The only remaining possibility is (iv), but in this case we may once again extend our embedding of G'' to an embedding of G. $\square$