9 Higher Surfaces

Embeddings in Other Surfaces

Disc: Any space which can be continuously deformed to $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$.

Surfaces and Embeddings: A *surface* is a topological space with the property that every point has a neighborhood which is a disc (so locally, it looks like the plane). The definition of graph embedding in the plane extends naturally to *embeddings* in other surfaces.

Sphere: We define the *sphere* to be $\mathcal{S} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$.

Observation 9.1 *The following are equivalent for every graph G.*

(i) *G is planar.*

(ii) *G has an embedding in the sphere.*

(iii) *G has an embedding in a disc.*

Torus: The *torus* is a surface which is obtained from the square

$$\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$$

by identifying the points $(0, y)$ and $(1, y)$ for every $0 \leq y \leq 1$ and identifying $(x, 0)$ and $(x, 1)$ for every $0 \leq x \leq 1$.

Handles: To add a handle to a surface S, we remove two disjoint discs from it, and then add a cylinder, so that each end of the cylinder is identified with the boundary of (a distinct) one of the removed discs.

Genus: For every nonnegative integer g, we let $\mathcal{S}_g$ denote a surface obtained from $\mathcal{S}$ by adding g handles. There is a theorem which states that any two surfaces obtained in this manner are topologically equivalent (homeomorphic), and we call such a space the *surface of genus g*. Note that $\mathcal{S}_1$ is equivalent to the torus.

Observation 9.2 *For every graph G there exists g so that G has an embedding in $\mathcal{S}_g$.*

Proof: Draw G in the plane (possibly with crossings). Then, anytime two edges cross, add a handle near this crossing point, and route one edge over the other. □

2-Cell: An embedding of G in a surface is a 2-*cell* embedding if every face is a disc (faces are defined analogously with planar embeddings).

Theorem 9.3 *Let G be a one vertex graph 2-cell embedded in $\mathcal{S}_g$ so that there is exactly one face. Then $|E(G)| = 2g$.*

Proof: omitted.

Theorem 9.4 (Euler's Formula) *If G is a connected graph 2-cell embedded in $\mathcal{S}_g$ then*

$$|V(G)| - |E(G)| + |F(G)| = 2 - 2g$$

Proof: We proceed by induction on $|E(G)|$. If there is a non-loop edge e, then the result follows by applying induction to $G \cdot e$. Otherwise, every edge is a loop. If there are at least two faces, we may choose a loop edge e with distinct faces on either side and then the result follows by applying induction to $G - e$. If no such edge exists, then the result follows by the above theorem. □

Theorem 9.5 (Heawood's Theorem) *If G is a loopless graph which can be embedded in $\mathcal{S}_g$, with $g > 0$ then $\chi(G) \leq \frac{7+\sqrt{1+48g}}{2}$.*

Proof: Set $c = \frac{7+\sqrt{1+48g}}{2}$. By Observation 6.2, it suffices to show that every simple graph embedded in $\mathcal{S}_g$ has a vertex of degree $\leq c - 1$. Suppose (for a contradiction) that G is such a graph with $\delta(G) \geq c$. Note that this implies $|V(G)| \geq c$ and note as well that every face has size ≥ 3 so $3|F(G)| \leq 2|E(G)|$. In the equation below, we use these facts with Euler's

Formula.

$$\begin{aligned} c(c-7) &= 12g - 12 \\ &= -6|V(G)| + 6|E(G)| - 6|F(G)| \\ &\geq -6|V(G)| + 2|E(G)| \\ &= \sum_{v \in V(G)} (deg(v) - 6) \\ &\geq |V(G)|(c-6) \\ &\geq c(c-6) \end{aligned}$$

Since $c \geq 7$ by definition, this is contradictory. □

Corollary 9.6 *Every graph which can be embedded in a torus has chromatic number* ≤ 7 *and this bound is best possible.*

Proof: The upper bound is a consequence of Heawood's Theorem. To see that this is the best possible upper bound, observe that K_7 may be embedded in the torus as in the figure below.