TORUS ACTIONS AND COMBINATORICS

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Abstract. These are lecture notes for a "Tapas" course given at the Fields Institute in November, 2017.

1. Introduction

Definition 1.1. A *complexity k T-variety* is a normal variety X equipped with an effective action

$$T \times X \to X$$

where $T \cong (\mathbb{K}^*)^m$ and $k = \dim X - \dim T$.

Examples include:

- $k = 0$: toric varieties. Everything is determined through combinatorics.
- $k = \dim X$: normal varieties with no additional structure.
- $X = G/P$ with T a maximal subtorus of G, e.g. $G(2,4)$ has a $T = (\mathbb{K}^*)^3$-action, hence is a complexity-one T-variety.
- Toric vector bundles $\mathcal{E}$ over a toric variety X (studied by Klyachko, Payne, Hering, Smith, Di Rocco, etc).
- $X \hookrightarrow Y$ toric varieties with $X = V(f)$, f a homogeneous binomial of degree u. Perturbations of f lead to complexity-one T-varieties, where $T = u^\perp$ (studied by Altmann).

The *goal* of these lectures is to develop a pseudo combinatorial language to study T-varieties. This is the theory of p-divisors introduced by Altmann and Hausen.

Application I hope to cover:

- Rank two toric vector bundles are Mori Dream Spaces (first proven independently by J. Gonzalez and Hausen–Süß).

2. Motivation and Classical Cases

Notation: N and M are mutually dual lattices, $T = \operatorname{Spec} \mathbb{K}[M] \cong N \otimes \mathbb{K}^*$.

2.1. Affine toric varieties.

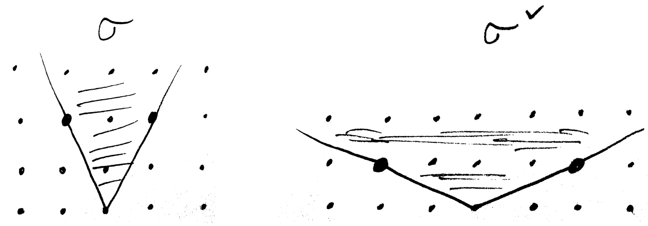
For σ a rational polyhedral cone in $N_\mathbb{R}$,

$$\sigma^\vee = \{u \in M_\mathbb{R} \mid \langle v, u\rangle \geq 0 \ \forall\ v \in \sigma\}$$

$$X(\sigma) = \operatorname{Spec} \mathbb{K}[\sigma^\vee \cap M]$$

is a normal toric variety of dimension equal to rank N.

Example 2.1. For



we have the variety given by

$$\operatorname{rank}\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 \end{pmatrix} \leq 1$$

2.2. **Good $\mathbb{K}^*$ actions.** In this situation, X is a normal affine variety, $T = \mathbb{K}^*$ with $X^T = \operatorname{Spec} \mathbb{K}$, or equivalently, $H^0(X, \mathcal{O}_X)^T = \mathbb{K}$. These varieties have been studied extensively by Demazure, Pinkham, and others.

Construction: Y a projective variety, D an ample $\mathbb{Q}$-divisor with Cartier multiple $\rightsquigarrow$

$$X = \operatorname{Spec} \bigoplus_{k \in \mathbb{Z}_{\geq 0}} H^0(Y, \mathcal{O}(k \cdot D)) \cdot \chi^k.$$

Theorem 2.2. *X is a normal $\mathbb{K}^*$-variety of dimension* $\dim Y + 1$.

Example 2.3. Taking for example

$$Y = \mathbb{P}^1 \qquad D = \frac{-1}{2} \cdot (\{0\} + \{1\}) + \frac{3}{2} \cdot \{\infty\}$$

leads to

$$\bigoplus H^0(\mathbb{P}^1, \mathcal{O}(kD)) \cdot \chi^k = \mathbb{K}[y(y-1)\chi^2, y^2(y-1)\chi^2, y^2(y-1)^2\chi^3]$$

so X is cut out by $c^2 = ab(b-a)$, i.e. a singularity of type D_4.

Exercise 2.4. Construct Y, D for other simple singularities, e.g. A_n, D_n, E_6, E_7, E_8.

Partial proof sketch for Theorem 2.2. Set

$$A = \bigoplus A_k = \bigoplus H^0(\mathcal{O}(k \cdot D)).$$

Let m be such that mD is very ample $\rightsquigarrow$

$$\widetilde{A} = \bigoplus A_{mk}$$

is the normalization of the coordinate ring of $Y \hookrightarrow \mathbb{P}^n$ via linear system $|mD|$.

Lemma 2.5. *A is integrally closed.*

Proof. It suffices to consider homogeneous elements in $Q(A)$ with homogeneous integral equations. For $f \in Q(A)$ integral over A of degree k, f^m will be integral over $\widetilde{A}$, which we know is integrally closed. Hence, $f^m \in A_{mk}$. Note

$$s \in A_k \iff \nu_P(s) + kc_P \geq 0$$

where $D = \sum c_P \cdot P$. Thus,

$$\begin{aligned} f^m \in A_{km} &\iff \\ \nu_P(f^m) + kmc_P \geq 0 &\iff \\ \nu_P(f) + mc_P \geq 0 &\iff \\ f \in A_k. \end{aligned}$$

□

Lemma 2.6. *A is integral over $\widetilde{A}$.*

Proof. $f \in A_k$ implies $f^m \in A_{mk} \subset \widetilde{A}$. □

Now, for finite generation of A, find $s \in A$ such that $Q(\widetilde{A}[s]) = Q(A)$ (any s in degree relatively prime to m suffices). Then A is the normalization of $\widetilde{A}[s]$, hence finite over $\widetilde{A}$, hence finitely generated.

□

2.3. **Torsors.** Consider a T-variety X with free T-action. In this situation, there is a good quotient $\pi : X \to Y = X/T$, and X comes with local trivializations $X_{|U} \cong U \times \operatorname{Spec} \mathbb{K}[M]$ compatible with the T-action, for $U \subset Y$.

For any $u \in M$, we get a line bundle $\mathcal{L}(u) = \pi_*(\mathcal{O}_X)_u$ on Y: using the above trivializations,

$$\mathcal{L}(u)_{|U} \cong \mathcal{O}_U \cdot \chi^u.$$

Locally, we check

Lemma 2.7.

$$\mathcal{L}(u) \otimes \mathcal{L}(w) = \mathcal{L}(u + w)$$

and

$$X = \operatorname{Spec}_Y \bigoplus_{u \in M} \mathcal{L}(u).$$

Choosing a basis $e_1, \ldots, e_m$ of M leads to line bundles $\mathcal{L}_i(e_i)$ encoding the information of X. Up to isomorphism, X only depends on the classes of the $\mathcal{L}_i$. Choosing Cartier divisors D_i such that $\mathcal{L}_i \cong \mathcal{O}(D_i)$ we obtain a linear map

$$\begin{aligned} \mathcal{D} : M &\to \operatorname{CaDiv} Y \\ u &\mapsto \sum u_i D_i \end{aligned}$$

encoding X up to isomorphism. If $D_i = \sum c_{P,i} P$ where P are prime divisors, we can rewrite

$$\mathcal{D} = \sum v_P P$$

for $v_P = \sum c_{P,i} e_i^*$. Here, we are thinking of $\mathcal{D}$ as a linear map via

$$\mathcal{D}(u) = \sum \langle v_P, u \rangle \cdot P.$$

3. Polyhedral Divisors

The theory of polyhedral divisors presented here is due to Altmann and Hausen [AH06]. Another reference is the survey [AIP+12]. This theory combines aspects of the three classical situations above.

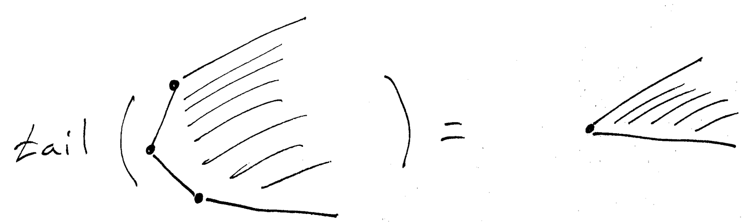
Setup: σ is a pointed cone in $N_{\mathbb{R}}$, Y is a normal semiprojective (projective over something affine) variety.

Definition 3.1. A polyhedral divisor on Y with tailcone σ is a "finite" formal sum

$$\mathcal{D} = \sum_{P \subset Y} \mathcal{D}_P \cdot P$$

where $P \subset Y$ are prime divisors, $\mathcal{D}_P$ are polytopes in $N_{\mathbb{R}}$ with tailcone σ, and finite means that $\mathcal{D}_P = \sigma$ for all but finitely many P.

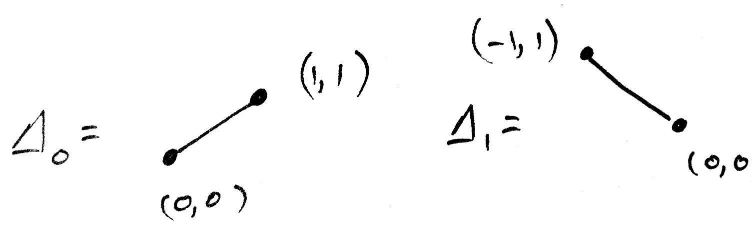
The tailcone of a polyhedron is its cone of unbounded directions:



Example 3.2. Take $Y = \mathbb{A}^2 = \operatorname{Spec} \mathbb{K}[x, y]$, $\sigma = 0$,

$$\Delta = \Delta_0 \cdot V(y) + \Delta_1 \cdot V(x - y)$$

with

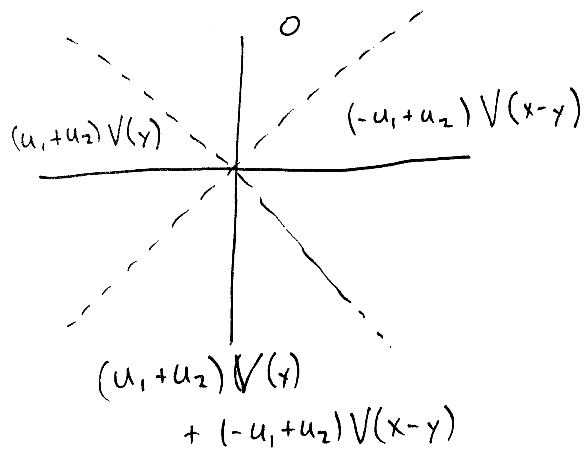


Polyhedral divisors determine piecewise linear maps

$$\begin{aligned} \sigma^\vee &\to \operatorname{Div}_{\mathbb{Q}} Y \\ u &\mapsto \sum_P \min\langle \mathcal{D}_P, u \rangle \cdot P. \end{aligned}$$

- All but finitely many coefficients are 0.
- Well defined, since $\operatorname{tail}(\mathcal{D}_P) = \sigma$.

Compare this to the linear map from the torsor example.



Example 3.3.

The map $\mathcal{D}$ has an important property — it is *convex*:

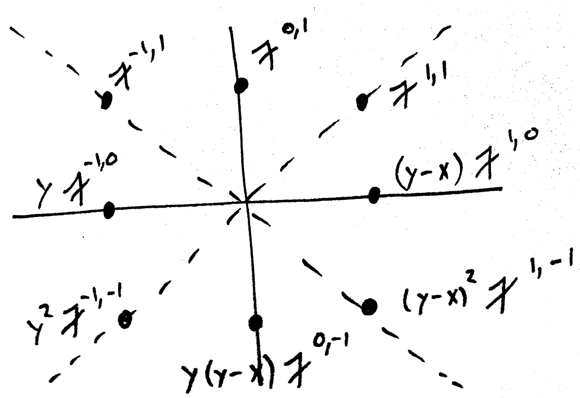
$$\mathcal{D}(u) + \mathcal{D}(v) \leq \mathcal{D}(u + v).$$

Hence, we can construct a sheaf of $\mathcal{O}_Y$-algebras and associated schemes:

$$\mathcal{O}(\mathcal{D}) := \bigoplus_{u \in \sigma^\vee \cap M} \mathcal{O}(\mathcal{D}(u)) \cdot \chi^u$$

$$\widetilde{X}(\mathcal{D}) := \operatorname{Spec}_Y \mathcal{O}(\mathcal{D})$$

$$X(\mathcal{D}) := \operatorname{Spec} H^0(Y, \mathcal{O}(\mathcal{D})).$$



Example 3.4.

We would like for $X(\mathcal{D})$ to be a T-variety (whose dimension we can predict). For this, we need additional criteria:

Definition 3.5. $\mathcal{D}$ is *proper* or a *p-divisor* if

- For all $u \in \sigma^\vee \cap M$, $\mathcal{D}(u)$ is $\mathbb{Q}$-Cartier and semiample, i.e. has a basepoint free multiple.
- For all $u \in (\sigma^\vee \cap M)^\circ$, $\mathcal{D}(u)$ is big, i.e. has a multiple admitting a section with affine complement.

Exercise 3.6. If Y is a curve, the $\mathcal{D}$ is proper if and only Y is affine, or $\deg \mathcal{D} \subsetneq \sigma$ and for all $u \in M \cap \sigma^\vee$ with $u^\perp \cap \deg \mathcal{D} \neq \emptyset$, $\mathcal{D}(u)$ has a principal multiple. Here,

$$\deg \mathcal{D} = \sum_P \mathcal{D}_P.$$

Theorem 3.7. [AH06] *We have the following:*

(1) $X(\mathcal{D})$, $\widetilde{X}(\mathcal{D})$ *are* T*-varieties of dimension* $\dim Y + \operatorname{rank} M$*.*
(2) $\pi : \widetilde{X}(\mathcal{D}) \to Y$ *is a good quotient*[1]*.*
(3) $\phi : \widetilde{X}(\mathcal{D}) \to X(\mathcal{D})$ *is proper and birational.*

Proof sketch. First assume "easy" situation: σ the positive orthant, $\mathcal{D}$ linear with $\mathcal{D}(u)$ Cartier $\rightsquigarrow$

$$\mathcal{E} = \bigoplus_i \mathcal{O}(\mathcal{D}(e_i))$$

is a vector bundle. Then

$$H^0(Y, \mathcal{E}) = H^0(\mathbb{P}(\mathcal{E}), \bigoplus_{k \geq 0} \mathcal{O}_{\mathbb{P}(\mathcal{E})}(k))$$

and we can reduce to the simply graded case. Note that $\mathcal{D}$ proper $\implies$ $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ semiample.

For general situation, subdivide $\sigma^\vee$ into simplices ω_i on which $\mathcal{D}$ is linear; coarsening the lattice gives a finite cover of something in the above "easy" situation. □

Theorem 3.8 ([AH06])**.** *Every affine* T*-variety can be constructed as above.*

We'll see part of a constructive proof of this below. The proof in [AH06] makes use of GIT, which we will omit. However, I find the following alternative argument conceptually useful, albeit incomplete:

- Let X° be the open subset of X with finite stabilizers.
- X° has (non-separated) quotient Y.
- Over Y, X° is "almost" a torsor; can be represented by a linear $\mathcal{D} : M \to \mathrm{CaDiv}_{\mathbb{Q}}(Y)$ (think of stack quotients and the root construction).
- Not all sections of $\mathcal{O}_{X^\circ}$ extend to X; this is governed by weight cone ω:

$$X = \bigoplus_{u \in \omega \cap M} H^0(Y, \mathcal{O}(\mathcal{D}(u))).$$

- Y is in general not separated; replacing Y by a separation leads to coefficients of $\mathcal{D}$ which are more general polyhedra, not just translates of $\omega^\vee$.

What is missing from this argument is that it is not entirely clear how to ensure that the resulting polyhedral divisor is proper.

The polyhedral divisor $\mathcal{D}$ encodes the fibers of $\pi : \widetilde{X}(\mathcal{D}) \to Y$, which are (unions of) toric varieties. We'll see a few more details on this later.

[1] i.e. an affine morphism locally given by taking invariants

4. Toric Downgrades, Existence, and Uniqueness

4.1. **Toric Downgrades.** Let $X = X(\sigma)$ be an affine toric variety for a cone σ in $\widetilde{N}_{\mathbb{R}}$. For a subtorus T of $\widetilde{T} = \widetilde{N} \otimes \mathbb{K}^*$, how do we describe X as a T-variety?

The inclusion $T \hookrightarrow \widetilde{T}$ corresponds to a surjection $\widetilde{M} \to M$. This leads to the following dual exact sequences:

$$0 \longrightarrow N \underset{s}{\overset{p}{\rightleftarrows}} \widetilde{N} \overset{q}{\longrightarrow} \overline{N} \longrightarrow 0$$

$$0 \longleftarrow M \overset{p^*}{\longleftarrow} \widetilde{M} \underset{t}{\overset{q^*}{\leftrightarrows}} \overline{M} \longleftarrow 0$$

Here we have chosen a cosection s of p. This determines t via $t(u) = u - s^*(p^*(u))$, viewed as an element of $\overline{M}$.

For $u \in M$, the degree u piece of $A = H^0(X, \mathcal{O}_X)$ is

$$A_u = \bigoplus_{w \in (p^*)^{-1}(u) \cap \sigma^\vee \cap M} \chi^w$$

Setting

$$\Delta(u) = t((p^*)^{-1}(u) \cap \sigma^\vee)$$

we obtain

$$A_u \cong \bigoplus_{w \in \Delta(u) \cap \overline{M}} \chi^w$$

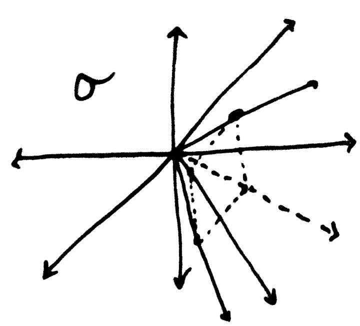
with these isomorphisms compatible with the ring structure (since t is additive)! As u ranges over M, only finitely many normal fans $\Sigma(\Delta(u))$ appear; let Σ be their coarsest common refinement. The $\Delta(u)$ represent semiample $\mathbb{Q}$-divisors on $X(\Sigma)$, and the map

$$u \mapsto \Delta(u)$$

is piecewise linear and convex, hence corresponds to a p-divisor $\mathcal{D}$ on $X(\Sigma)$.[2]

Example 4.1. Consider the cone σ whose rays are generated by the columns of

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

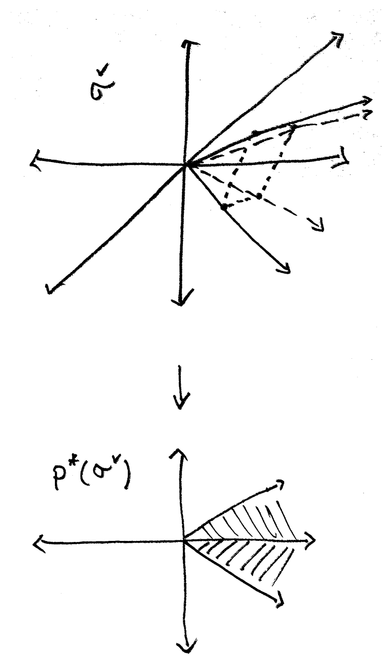


[2]The bigness criterion must also be checked, but is not difficult.

Then $\sigma^\vee$ has rays generated by the columns of

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

We can consider the subtorus $T = (0,0,1)^\perp$. We take the obvious choices for s and t.



Setting

$$\omega_1 = \mathbb{R}_{\geq 0}\{(1,0),(1,1)\}$$
$$\omega_2 = \mathbb{R}_{\geq 0}\{(1,0),(1,-1)\}$$

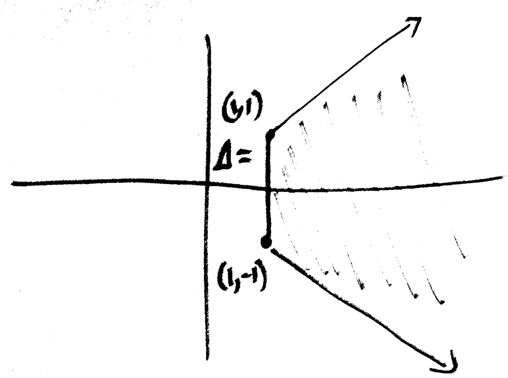
we obtain

$$\Delta(u) = \begin{cases} [u_2 - u_1, u_1 - u_2] & u \in \omega_1 \\ [-u_1 - u_2, u_1 + u_2] & u \in \omega_2. \end{cases}$$

This leads to $Y = \mathbb{P}^1$ and

$$\mathcal{D} = \Delta \cdot (\{0\} + \{\infty\})$$

for



Exercise 4.2. Show that Σ is the fan obtained by considering the coarsest common refinement of the cones $p(\tau)$, where τ ranges over all faces of σ. Show that we can represent $\mathcal{D}$ as

$$\mathcal{D} = \sum_{\rho \in \Sigma^{(1)}} s(q^{-1}(v_\rho) \cap \sigma) \cdot D_\rho$$

where v_ρ is the primitive generator of a ray ρ, and D_ρ the corresponding prime invariant divisor on $X(\Sigma)$.

4.2. **General Affine T-Varieties.** We can adapt the above setup to construct a p-divisor for any affine T-variety: If X is equivariantly embedded in $\mathbb{A}^n$,[3] this gives in particular an embedding

$$T \hookrightarrow (\mathbb{K}^*)^n = \widetilde{T}.$$

The above construction produces a quotient $X(\Sigma)$ with p-divisor $\mathcal{D}$ describing $\mathbb{A}^n$ as a T-variety.

Possibly passing to a smaller affine space, we can assume that $X^\circ := X \cap \widetilde{T} \neq \emptyset$. Let Z be the closure of the image of X° in $X(\Sigma)$, and $\phi : Y \to Z$ its normalization.

Proposition 4.3. *We have that*

$$\mathcal{D}' = \phi^*(\mathcal{D}_{|Z})$$

is a p-divisor, and

$$X(\mathcal{D}') = X.$$

Proof. The claim that $\mathcal{D}'$ is a p-divisor is straightforward. For the second claim, one needs to use a bit of GIT [AH06]. □

4.3. **Uniqueness.** How much choice does one have in representing a T-variety X by a p-divisor $\mathcal{D}$? The answer given by [AH06] is that $\mathcal{D}$ is uniquely determined up to three different kinds of equivalences:

- For $\phi : Y' \to Y$ proper and birational, $\mathcal{D}$ a p-divisor on Y, $\phi^*(\mathcal{D})$ is the p-divisor given by $\phi^*(\mathcal{D})(u) = \phi^*(\mathcal{D}(u))$. Then we have $X(\mathcal{D}) \cong X(\phi^*(\mathcal{D}))$, since sections don't change under ϕ^*.
- For a lattice automorphism $\rho \in \mathrm{Aut}(M)$,

$$\rho(\mathcal{D}) = \sum \rho(\mathcal{D}_P) \cdot P$$

is a p-divisor on Y, and $X(\mathcal{D}) \cong X(\rho(\mathcal{D}))$.

[3]This is possible by Sumihiro's theorem.

- For $\mathfrak{f} \in N \otimes \mathbb{K}(Y)^*$, let $\operatorname{div}\mathfrak{f}$ be the principal polyhedral divisor given by

$$(\operatorname{div}\mathfrak{f})(u) = \operatorname{div}(\mathfrak{f}(u)).$$

  Then $X(\mathcal{D}) \cong X(\mathcal{D} + \operatorname{div}\mathfrak{f})$, with isomorphism induced by $\chi^u \mapsto \mathfrak{f}(u)^{-1}\chi^u$.

It is a theorem of [AH06] that, up to these three equivalences, the p-divisor for X is uniquely determined. Moreover, one can similarly define a category of p-divisors, and show that it is equivalent to the category of normal affine varieties with torus action.

5. The Non-Affine Case

5.1. **Divisorial Fans.** We now wish to globalize our combinatorial description of T-varieties to the non-affine case. This has been done in [AHS08]. The basic idea is to glue $X(\mathcal{D})$ together for some nice set $\mathcal{S}$ of polyhedral divisors.

A necessary change: we now allow $\emptyset$ as a coefficient for a polyhedral divisor $\mathcal{D}$. For $\mathcal{D}$ on Y,

$$\operatorname{Loc}\mathcal{D} = Y \setminus \bigcup_{P \mid \mathcal{D}_P = \emptyset} P.$$

The polyhedral divisor $\mathcal{D}$ restricts to a "usual" polyhedral divisor $\mathcal{D}_{|\operatorname{Loc}\mathcal{D}}$, and is *proper* exactly when this restriction is.

Example 5.1. For Y a projective curve, $\mathcal{D}$ a p-divisor on Y, $\deg\mathcal{D} = \emptyset \iff \operatorname{Loc}\mathcal{D}$ is affine.

Let $\mathcal{D}, \mathcal{D}'$ be polyhedral divisors on Y.

- $\mathcal{D} \subset \mathcal{D}' \iff \mathcal{D}_P \subset \mathcal{D}'_P$ for all prime divisors P.
- $\mathcal{D} \cap \mathcal{D}' = \sum_P \mathcal{D}_P \cap \mathcal{D}'_P \cdot P$

Note that

$$\mathcal{D} \subset \mathcal{D}' \implies \mathcal{D}(u) \geq \mathcal{D}'(u) \implies \mathcal{O}(\mathcal{D}') \subset \mathcal{O}(\mathcal{D})$$

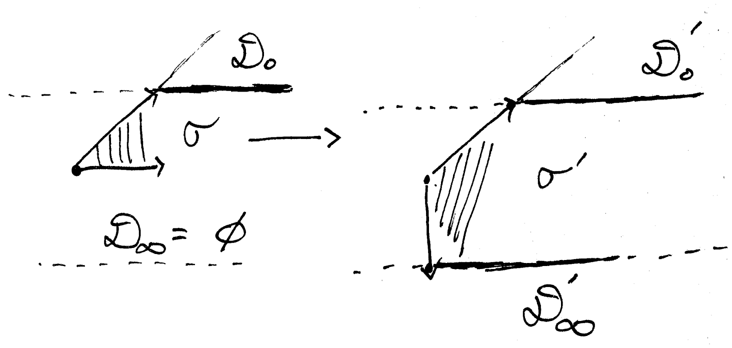
in which case we obtain a dominant morphism

$$X(\mathcal{D}) \to X(\mathcal{D}').$$

Definition 5.2. $\mathcal{D}$ is a face of $\mathcal{D}'$ ($\mathcal{D} \prec \mathcal{D}'$) if and only if this morphism is an open embedding.

Remark 5.3. $\mathcal{D} \prec \mathcal{D}'$ implies that for all P, $\mathcal{D}_P$ is a face of $\mathcal{D}'_P$ ($\mathcal{D}_P \prec \mathcal{D}'_P$). However, the converse is not true.

Example 5.4. Consider $\mathcal{D}$ and $\mathcal{D}'$ as pictured. This describes a downgrade of the dominant toric morphism $\mathbb{A}^2 \to \mathbb{A}^2$ coming from blowing up $\mathbb{A}^2$ at the origin. Despite $\mathcal{D}_P \prec \mathcal{D}'_P$ for all P, this is not an open embedding.



There is an explicit (and ugly criterion) for in general when $\mathcal{D} \prec \mathcal{D}'$. In complexity-one, it simplifies:

Proposition 5.5 ([IS11]). *On a projective curve Y, $\mathcal{D} \prec \mathcal{D}' \iff \deg \mathcal{D} = \operatorname{tail} \mathcal{D} \cap \deg \mathcal{D}'$.*

Definition 5.6. A *divisorial fan* on Y is a finite set $\mathcal{S}$ of p-divisors, closed under intersection, such that for all $\mathcal{D}, \mathcal{D}' \in \mathcal{S}$,

$$\mathcal{D} \cap \mathcal{D}' \prec \mathcal{D}, \mathcal{D}'.$$

Construction:

$$X(\mathcal{S}) = \coprod_{\mathcal{D} \in \mathcal{S}} X(\mathcal{D}) / \sim$$

where glueing is done along

$$X(\mathcal{D}) \hookleftarrow X(\mathcal{D} \cap \mathcal{D}') \hookrightarrow X(\mathcal{D}').$$

Theorem 5.7. [AHS08] *Every T-variety can be constructed in this way.*

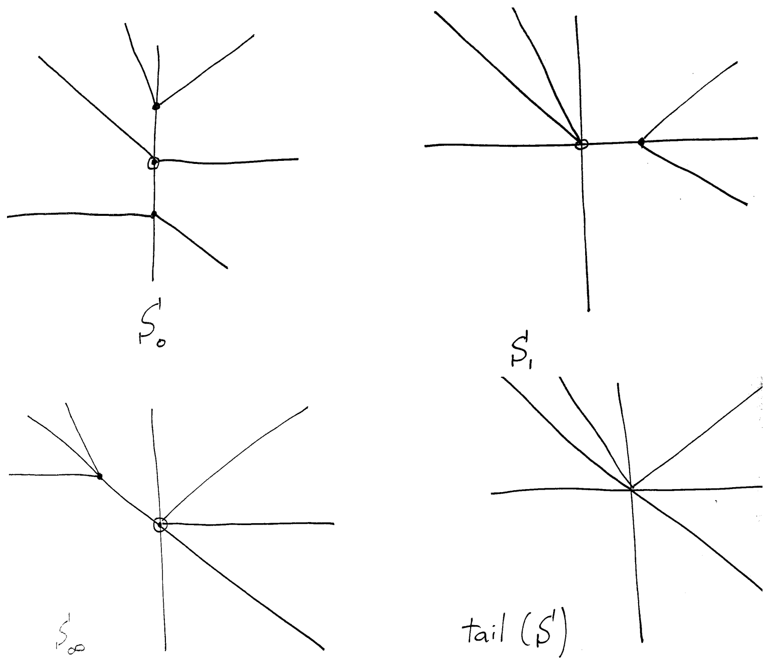
Proof. Sumihiro's theorem plus valuative criterion for separatedness. □

Remark 5.8. For all P,

$$\mathcal{S}_P = \{\mathcal{D}_P \mid \mathcal{D} \in \mathcal{S}\}$$

forms a polyhedral complex called the *slice* of $\mathcal{S}$ at P.

Example 5.9. $X = \mathbb{P}(\Omega_{\mathcal{F}_1})$.



We need to know which coefficients belong to a common p-divisor. Here, all full-dimensional coefficients with common tailcone belong together.

5.2. **Divisors and Global Sections.** Here we follow [PS11]. The setup is a divisorial fan $\mathcal{S}$ on Y, leading to

$$\begin{array}{ccc} \widetilde{X}(\mathcal{S}) & \xrightarrow{\phi} & X(\mathcal{S}) \\ \big\downarrow{\scriptstyle \pi} & & \\ Y & & \end{array}$$

T-invariant prime divisors on $\widetilde{X}(\mathcal{S})$ come in two types:

- *Vertical*: components of $\pi^{-1}(P)$ for prime $P \subset Y$. Correspond to vertices $v \in \mathcal{S}_P \rightsquigarrow D_{P,v}$.
- *Horizontal*: image covers all of Y. Correspond to rays ρ of $\mathrm{tail}(\mathcal{S}) = \mathcal{S}_\eta$, for $\eta \in Y$ a general point $\rightsquigarrow D_\rho$.

To see this, one can take a log resolution of Y, after which $\widetilde{X}(\mathcal{S}) \to Y$ comes étale locally from a toric downgrade with fan Σ. Rays of Σ in "height zero" give rays of $\mathrm{tail}\,\mathcal{S}$, whereas other rays give vertices in slices of $\mathcal{S}$.

The prime invariant divisors of $X(\mathcal{S})$ are all images of prime invariant divisors of $X(\mathcal{S})$. In general, some of these get contracted by ϕ. In the case of complexity one, only divisors of the form D_ρ may be contracted.

Proposition 5.10. *Given* $f \in \mathbb{K}(Y)$*,*

$$\mathrm{div}(f \cdot \chi^u) = \sum_{P,v} \mu(v)(\nu_P(f) + \langle v, u\rangle) D_{P,v} + \sum_\rho \langle v_\rho, u\rangle D_\rho$$

where $\nu_P(f)$ is the order of vanishing of f along P, $\mu(v)$ is the smallest natural number such that $\mu(v)v \in N$, and v_ρ is the primitive lattice generator of ρ.

Proof. After assuming $\widetilde{X}(\mathcal{S}) \to Y$ is (locally) toric, this follows from the classical toric formula. □

This leads to a formula for global sections of divisors. Let $D = \sum a_{P,v}\mu(v)D_{p,v} + \sum a_\rho D_\rho$ be a T-invariant divisor on $X(\mathcal{S})$. Define

$$\begin{aligned} \square^D &= \{u \in M_{\mathbb{R}} \mid \langle v_\rho, u\rangle + a_\rho \geq 0\} \\ \Psi^D_P(u) &= \min_{v \in \mathcal{S}^{(0)}_P} (a_{P,v} + \langle v, u\rangle) \\ \Psi^D : \square^D &\to \operatorname{Div}_{\mathbb{Q}} Y \\ u &\mapsto \sum_P \Psi^D_P(u)P \end{aligned}$$

Proposition 5.11. *We have a graded isomorphism*

$$H^0(X(\mathcal{S}), \mathcal{O}(D)) = \bigoplus_{u \in \square^D \cap M} H^0(Y, \mathcal{O}(\Psi^D(u))).$$

Proof. Given $f \cdot \chi^u \in \mathbb{K}(X)$, the D_ρ coefficient of $\operatorname{div}(f) + D$ is non-negative if and only if $\langle v_\rho, u\rangle + a_\rho \geq 0$. The $D_{P,v}$ coefficient is non-negative if and only if $\langle v, u\rangle + \nu_P(f) + a_{P,v} \geq 0$. □

Example 5.12. We consider the divisor

$$D = D_{0,(0,0)} + D_{0,(0,1)} + D_{0,(0,-1)} + D_{1,(0,0)} + D_{1,(1,0)}.$$

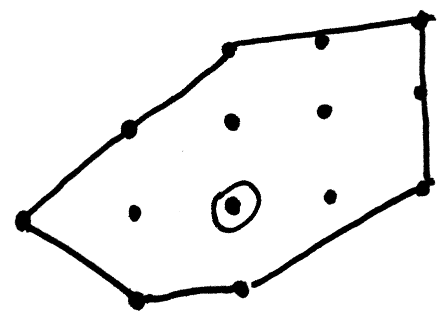
This is an anticanonical divisor on this variety. This leads to

$$\begin{aligned} \Psi^D_0(u) &= \begin{cases} 1 - u_2 & u_2 \geq 0 \\ 1 + u_2 & u_2 \leq 0 \end{cases} \\ \Psi^D_1(u) &= \begin{cases} 1 & u_1 \geq 0 \\ 1 + u_1 & u_1 \leq 0 \end{cases} \\ \Psi^D_\infty(u) &= \begin{cases} 0 & u_1 \leq u_2 \\ u_2 - u_1 & u_2 \leq u_1 \end{cases} \end{aligned}$$

and e.g.

$$H^0(\mathcal{O}(D))_{(-2,0)} \cong H^0(\mathbb{P}^1, \mathcal{O}(\{0\} - \{1\})).$$

The weights in which H^0 is supported are pictured below:



6. Cox Rings

Let X be a smooth projective variety with $\mathrm{Pic}(X) \cong \mathbb{Z}^n$. After choosing representatives $D_1, \ldots, D_n \in \mathrm{Div}(X)$ of a basis of $\mathrm{Pic}(X)$, we can define the Cox ring of X to be

$$\mathrm{Cox}(X) = \bigoplus_{u \in \mathbb{Z}^n} H^0(X, \mathcal{O}(u_1 D_1 + \ldots + u_n D_n))$$

with multiplication given by the natural multiplication of sections.[4] X is a *Mori Dream Space* (MDS) if $\mathrm{Cox}(X)$ is finitely generated.

We will apply the tools we have developed to prove that smooth, rational complexity-one T-varieties X are MDS. In particular, if $\mathcal{E}$ is a rank two toric vector bundle over a smooth toric variety, $\mathbb{P}(\mathcal{E})$ is MDS. The first statement was originally proven in a much more general setting in [HS10]; the latter statement was proven independently by [Gon12]. The arguments we are using are from [IV13]; similar arguments are used in [AP12] to also include the singular case with torsion in the class group.

We fix our rational complexity-one T-variety $X(\mathcal{S})$; here we have $Y = \mathbb{P}^1$. We can choose the representatives $D_1, \ldots, D_n$ to be T-invariant. Set $D(u) = \sum u_i D_i$. Then we obtain

$$\begin{aligned} \mathrm{Cox}(X) &= \bigoplus_{u \in \mathbb{Z}^n} H^0(X, D(u)) \\ &= \bigoplus_{u \in \mathbb{Z}^n} \bigoplus_{w \in M \cap \Box^{D(u)}} H^0(\mathbb{P}^1, \mathcal{O}(\Psi(u))) \quad = \bigoplus_{(u,w) \in \omega} H^0(\mathbb{P}^1, \mathcal{O}(\Psi^{D(u)}(w))) \end{aligned}$$

where ω is the cone in $(\mathbb{Z}^n \times M)_{\mathbb{R}}$ generated by $(u, \Box^{D(u)})$. The map

$$\begin{aligned} \mathcal{D} : \omega &\to \mathrm{Div}_{\mathbb{Q}}(\mathbb{P}^1) \\ (u, w) &\mapsto \Psi^{D(u)}(w) \end{aligned}$$

is piecewise-linear and convex, hence can be thought of as a polyhedral divisor! The one difficulty is that it might not be proper.

However, we can restrict $\mathcal{D}$ to the closed subcone ω' of ω on with the degree of $\mathcal{D}$ is non-negative, since elsewhere $\mathcal{O}(\mathcal{D})$ has no sections. Since $Y = \mathbb{P}^1$, this is already enough to guarantee that $\mathcal{D}(u, w)$ is semiample for all $(u, w) \in \omega'$. This in turn guarantees finite generation of the Cox ring.

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[4]More generally, one can define a Cox ring for singular X as long as the class group is finitely generated.

Exercise 6.1. Given representatives $D_1, \dots, D_n$ as above, explicitly determine the coefficients of the polyhedral divisor $\mathcal{D}$ describing the Cox ring.

7. Deformations

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