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Exercises

1. Consider $q \in \mathbb{G}(1,3)$. Show that

$$\{p \in \mathbb{G}(1,3) \mid p \cap q \neq \emptyset\} = \mathbb{G}(1,3) \cap V(p_{01}q_{23} - p_{02}q_{13} + p_{03}q_{12} + p_{12}q_{03} - p_{13}q_{02} + p_{23}q_{01}).$$

Here, the p_{ij}, q_{ij} are Plücker coordinates. *Hint: consider the matrix*

$$\begin{pmatrix} x_{00} & x_{01} & x_{02} & x_{03} \\ x_{10} & x_{11} & x_{12} & x_{13} \\ y_{00} & y_{01} & y_{02} & y_{03} \\ y_{10} & y_{11} & y_{12} & y_{13} \end{pmatrix}$$

with p, q the rowspans of the first two and last two rows.

2. Use the previous problem to determine equations for $\mathbb{G}(1,3)$ in its Plücker embedding.

3. For generic lines L_1, L_2, L_3, L_4 in $\mathbb{P}^3$, how many lines L are there such that $L_i \cap L \neq \emptyset$ for all i? *Hint: use the first exercise to describe this set as a subvariety of* $\mathbb{G}(1,3)$. Reinterpret this answer in the degenerate situation when L_1, L_2 and L_3, L_4 are coplanar.

4. Describe $\mathbf{F}_1(X)$ for $X = V(xy - z^2) \subset \mathbb{P}^3$.

5. Let $X = V(x_0^4 + x_1^4 + \ldots + x_4^4)$. What is the *expected dimension* of $\mathbf{F}_1(X)$? Describe $\mathbf{F}_1(X)$.

6. Let X be a smooth quadric threefold in $\mathbb{P}^4$. What is the expected dimension of $\mathbf{F}_2(X)$? Show that $\mathbf{F}_2(X)$ is empty.

7. Let $\mathcal{A}$ be a finite subset of $\mathbb{Z}^n$. Show that the ideal of $X_{\mathcal{A}}$ is generated by

$$\Big\{ \prod_{u \in \mathcal{A}} x_u^{a_u} - \prod_{u \in \mathcal{A}} x_u^{b_u} \,\Big|\, \sum a_u = \sum b_u \text{ and } \sum a_u \cdot u = \sum b_u \cdot u \Big\}.$$

8. Let X be a cubic surface in $\mathbb{P}^3$ with ideal $I = \langle f \rangle$. Consider a line $L \subset X$ such that for every $x \in L$, X is smooth at x. Show that

$$\mathrm{Hom}_S(I_L, S/I_L)_0 = 0$$

where $I_L \subset S = \mathbb{K}[x_0, x_1, x_2, x_3]/I$ is the homogeneous ideal of L in X. *Hint: after coordinate transformation, you can assume that $I_L = \langle x_0, x_1 \rangle$.*

9. How many lines does a smooth intersection of two quadrics in $\mathbb{P}^4$ contain? Prove your answer. *Hint: consider a (singular) toric intersection of two quadrics.*