Short and Long Run Causality Measures

Jean-Marie Dufour
and
Abderrahim Taamouti
University of Montreal
October 22, 2005

1 Objective and motivation

1.1 Objective:

1. To measure the strength of a causality between two variables X and Y at any horizon $h \geq 1$.

2. To measure the INDIRECT causality transmitted by an auxiliary variable Z.

Existing causality measures [see for example Geweke (1982, 1984, *JASA*), Gouriéroux, Monfort, and Renault (1987, *Annales D'Économie et De Statistique*)] have been established only for the horizon 1 and fail to capture the indirect causal effects.

1.2 Motivations:

1. It is possible to have a causality between two variables X and Y at horizon h STRICTLY higher than 1 [see Dufour and Renault (1998, *Econometrica*)].

2. It is possible that a variable Y does not cause a variable X at horizon 1, but causes it at horizon $h > 1$ (Indirect causality transmitted by an auxiliary variable Z) [see Lütkepohl (1993), Dufour and Renault (1998)].

Example 1 (Indirect Effect)

$$\begin{pmatrix} X(t+1) \\ Y(t+1) \\ Z(t+1) \end{pmatrix} = \begin{bmatrix} 0.60 & 0.00 & 0.80 \\ 0.00 & 0.40 & 0.00 \\ 0.00 & 0.60 & 0.10 \end{bmatrix} \begin{pmatrix} X(t) \\ Y(t) \\ Z(t) \end{pmatrix} + \begin{pmatrix} \varepsilon_X(t+1) \\ \varepsilon_Y(t+1) \\ \varepsilon_Z(t+1) \end{pmatrix},$$

$$\Rightarrow \ X(t+1) = 0.6\, X(t) + 0.8\, Z(t) + \varepsilon_X(t+1).$$

Since the coefficient of $Y(t)$ is null, we can conclude that Y does not cause X at horizon 1 [Wiener (1956, *McGraw-Hill*) and Granger(1969, *Econometrica*)].

If we consider the above model at time $(t+2)$:

$$\begin{aligned} X(t+2) = \ & 0.36\, X(t) + 0.48 Y(t) \\ & + 0.56\, Z(t) + 0.6 \varepsilon_X(t+1) \\ & + 0.8 \varepsilon_Z(t+1) + \varepsilon_X(t+2). \end{aligned}$$

Now the coefficient of $Y(t)$ is equal to 0.48, then Y causes X at horizon 2. Here we are in presence of an indirect effect $(0.48 = 0.80 \times 0.60)$.

$$Y \overset{0.6}{\rightarrow} Z \overset{0.8}{\rightarrow} X$$

Conclusion: How can we measure this indirect effect?

2 General structure

2.1 Processes:

$$\{X(t) : t \in \mathbb{Z}\}, \quad \{Y(t) : t \in \mathbb{Z}\}, \ \{Z(t) : t \in \mathbb{Z}\}.$$

2.2 Information sets:

$$\begin{aligned} \underline{X}_t &= \{X(s), s \leq t\}, \\ \underline{Y}_t &= \{Y(s), s \leq t\}, \\ \underline{Z}_t &= \{Z(s), s \leq t\}, \\ I_t &= \underline{X}_t \cup \underline{Y}_t \cup \underline{Z}_t. \end{aligned}$$

2.3 The variance of the forecast error of $X(t+h)$ *based on the information set* A_t*, for* $A_t = I_t, \underline{X}_t \cup \underline{Z}_t$,

$$\sigma^2(X(t+h) \mid A_t).$$

Definition 1 *For* $h \geq 1$*, we say that* Y *does not cause* X *at horizon* h *given all elements of* I_t *except those of* Y*, denoted*

$Y \underset{h}{\nrightarrow} X \mid Z$*, if*

$$\sigma^2(X(t+h) \mid I_t - \underline{Y}_t) = \sigma^2(X(t+h) \mid I_t).$$

[see Dufour and Renault (1998).]

3 The idea

By analogy to Geweke (1982, 1984), we propose a causality measures at horizon h, for $h \geq 1$.

Principle: " Y causes X at horizon h" if:

$$\sigma^2(X(t+h) \mid I_t - \underline{Y}_t) > \sigma^2(X(t+h) \mid I_t).$$

If the difference between the two variances is "big", then one can conclude that the past of Y contains more information about $X(t+h)$, and that Y causes more X at horizon h.

Measuring the difference between the two variances is the same as measuring the strength of the causality from Y to X at horizon h.

4 Nonparametric and parametric causality measures

4.1 Nonparametric causality measures:

1. Measure of feedback at horizon h

Definition 2 *For $h \geq 1$, where h is a positive integer, a causality measure from Y to X at horizon h, called the INTENSITY of the causality from Y to X at horizon h, denoted $C(Y \underset{h}{\rightarrow} X | Z)$, is given by:*

$$C(Y \underset{h}{\rightarrow} X | Z) = \ln \left[\frac{\sigma^2(X(t+h) \mid I_t - \underline{Y}_t)}{\sigma^2(X(t+h) \mid I_t)} \right]. \tag{1}$$

Similarly we can define a causality measure from X to Y at horizon h, called the INTENSITY of the causality from X to Y at horizon h, denoted $C(X \underset{h}{\rightarrow} Y | Z)$, by:

$$C(X \underset{h}{\rightarrow} Y | Z) = \ln \left[\frac{\sigma^2(Y(t+h) \mid I_t - \underline{X}_t)}{\sigma^2(Y(t+h) \mid I_t)} \right]. \tag{2}$$

2. Measure of the instantaneous effect at horizon h

Definition 3 *For $h \geq 1$, where h is a positive integer, an instantaneous causality measure between Y and X at horizon h, called the INTENSITY of the instantaneous causality between Y and X at horizon h, denoted $C(X \underset{h}{\leftrightarrow} Y | Z)$, is given by:*

$$\mathrm{C}(X \underset{h}{\leftrightarrow} Y | Z) = \ln \left[\frac{\sigma^2(X(t+h) \mid I_t)\, \sigma^2(Y(t+h) \mid I_t)}{\sigma^2(X(t+h), Y(t+h) \mid I_t)} \right]. \tag{3}$$

Interpretation :

$$C(X \underset{h}{\leftrightarrow} Y | Z) = \ln \left[(1 - \rho^2(X(t+h), Y(t+h) \mid I_t))^{-1} \right],$$

$$\Rightarrow$$

$$\rho^2(X(t+h), Y(t+h) \mid I_t) = 1 - (\exp(C(X \underset{h}{\leftrightarrow} Y | Z)))^{-1}.$$

$$\rho^2(X(t+h), Y(t+h) \mid I_t) \nearrow \Longleftrightarrow C(X \underset{h}{\leftrightarrow} Y | Z) \nearrow$$

where ρ represents the correlation coefficient between X and Y at time $(t+h)$ given I_t.

3. Measure of the dependence at horizon h

Definition 4 *For $h \geq 1$, where h is a positive integer, a measure of dependence between X and Y at horizon h, called the INTENSITY of the dependence between X and Y at horizon h, denoted $C^{(h)}(X, Y \mid Z)$, is given by:*

$$\mathrm{C}^{(h)}(X, Y \mid Z) = \ln \left[\frac{\sigma^2(X(t+h)|I_t - \underline{Y}_t)\, \sigma^2(Y(t+h)|I_t - \underline{X}_t)}{\sigma^2(X(t+h), Y(t+h)|I_t)} \right].$$

It is very easy to prove the following decomposition:

$$C^{(h)}(X, Y \mid Z) = C(X \underset{h}{\rightarrow} Y|Z) + C(Y \underset{h}{\rightarrow} X|Z) + C(X \underset{h}{\leftrightarrow} Y|Z).$$

4.2 <u>Parametric causality measures</u>:

In this paper we show how to calculate a parametric causality measures in the context of stationary vector autoregressive and moving average model ($VARMA$).

The parametric causality measures depend on parameters of $VARMA$ model (or on parameters of its VMA representation).

5 Estimation and Inference

5.1 Estimation:

As we mentioned in the previous slide the parametric causality measures depend on the unknown parameters of $VARMA$ model. So an estimator of causality measure $C(Y \underset{h}{\longrightarrow} X|Z)$, denoted $\hat{C}(Y \underset{h}{\longrightarrow} X|Z)$, can be obtained by replacing the unknown parameters of $VARMA$ model by their estimates from a finite sample.

Under assumption 1 [see next subsection] we show the convergence of the estimator $\hat{C}(Y \underset{h}{\longrightarrow} X|Z)$.

Calculation of causality measures using simulations:

We propose a simple technique to evaluate $C(Y \underset{h}{\longrightarrow} X|Z)$.

Principle: It consists in simulating a large sample, example $T = 100000$,..., from the unconstrained model (the model of joint process (X, Y, Z)) whose parameters are supposed to be known or estimated from a real data set. Once the large sample is simulated we use it to estimate the parameters of the constrained model (the model of joint process (X, Z)).

Example: For bivariate $VAR(1)$ model

$$\begin{pmatrix} X(t+1) \\ Y(t+1) \end{pmatrix} = \begin{bmatrix} \pi_{XX} & \pi_{XY} \\ \pi_{YX} & \pi_{YY} \end{bmatrix} \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} + \begin{pmatrix} u_X(t+1) \\ u_Y(t+1) \end{pmatrix},$$

$$\mathrm{E}\Big[u(t)u(s)'\Big] = \begin{cases} \Sigma_u = \begin{pmatrix} \sigma^2_{u_X} & 0 \\ 0 & \sigma^2_{u_Y} \end{pmatrix} & \text{for } s = t, \\ 0 & \text{for } s \neq t. \end{cases},$$

we have:

$$\mathrm{C}(Y \underset{1}{\rightarrow} X) = \ln\Big[\frac{(1+\pi^2_{YY})\sigma^2_{u_X} + \pi^2_{XY}\sigma^2_{u_Y} - \sqrt{((1+\pi_{YY})^2\sigma^2_{u_X} + \pi^2_{XY}\sigma^2_{u_Y})^2 - 4\pi^2_{YY}\sigma^4_{u_X}}}{2\sigma^2_{u_X}}\Big],$$

$$\mathrm{C}(Y \underset{2}{\rightarrow} X) = \ln\Big[\frac{4\pi^2_{YY}\sigma^4_{u_X} + [(1+\pi^2_{YY})\sigma^2_{u_X} + \pi^2_{XY}\sigma^2_{u_Y} - \sqrt{((1+\pi_{YY})^2\sigma^2_{u_X} + \pi^2_{XY}\sigma^2_{u_Y})^2 - 4\pi^2_{YY}\sigma^4_{u_X}} - 2\pi_{YY}\sigma^2_{u_X}]^2}{((1+\pi^2_{XX})\sigma^2_{u_X} + \pi^2_{XY}\sigma^2_{u_Y})[(1+\pi^2_{YY})\sigma^2_{u_X} + \pi^2_{XY}\sigma^2_{u_Y} - \sqrt{((1+\pi_{YY})^2\sigma^2_{u_X} + \pi^2_{XY}\sigma^2_{u_Y})^2 - 4\pi^2_{YY}\sigma^4_{u_X}}]}\Big].$$

If we take

$$\begin{bmatrix} \pi_{XX} & \pi_{XY} \\ \pi_{YX} & \pi_{YY} \end{bmatrix} = \begin{bmatrix} 0.5 & 0.7 \\ 0.4 & 0.35 \end{bmatrix} \text{ and } \Sigma_u = I_2.$$

then we get the following theoretical values of $C(Y \underset{1}{\longrightarrow} X)$ and $C(Y \underset{2}{\longrightarrow} X)$,

$$C(Y \underset{1}{\longrightarrow} X) = 0.425, \; C(Y \underset{2}{\longrightarrow} X) = 0.197.$$

Using simulation technique:

1. We simulate a large sample of $T = 600,000$ observations from the unconstrained model (the model of (X, Y)), under assumption $u \sim N(0, I_2)$,

2. We use the simulated large sample to regress X on its own past,

3. We evaluate the causality measures at horizon 1 and 2 by calculating the constrained ($\sigma^2(X(t+h) \mid \underline{X}_t)$) and unconstrained ($\sigma^2(X(t+h) \mid \underline{X}_t, \underline{Y}_t)$) variances.

For different orders p of the constrained model we get:

p	$C(Y \underset{1}{\longrightarrow} X)$	$C(Y \underset{2}{\longrightarrow} X)$
1	0.519	0.567
2	0.430	0.220
3	0.427	0.200
4	0.425	0.199
5	0.426	0.198
10	0.425	0.197
15	0.426	0.199
20	0.425	0.197

5.2 Bootstrap confidence interval:

We suppose that $\{W(s) \equiv (X(s), Y(s), Z(s)), s \leq t\}$, follows a $VAR(p)$ model.

Assumption 1 :

1) All fourth moments of the error term in the constrained model (the model of the joint process (X, Z)) are finite.

2) The fitted order of the constrained model, denoted k, is chosen as a function of T such that $k^3/T \to 0$ as $k, T \to \infty$, where T represents the sample size.

3) k is chosen as a function of T such that $T^{1/2} \sum_{j=k+1}^{\infty} \| \pi_j^c \| \to 0$ as $k, T \to \infty$, where π_j^c, for $j = 1, ..., \infty$, represent the autoregressive coefficients of the constrained model.

Assumption 2 :

The function which associates the constrained parameters (the parameters of model of process (X, Z)) to the unconstrained parameters (the parameters of model of process (X, Y, Z)) is continuous and differentiable.

Asymptotic distribution of $\hat{C}(Y \underset{h}{\longrightarrow} X|Z)$ *:*

Under assumptions 1 and 2 we show that,

$$(T-p)^{1/2}(\hat{C}(Y \underset{h}{\longrightarrow} X|Z) - C(Y \underset{h}{\longrightarrow} X|Z)) \xrightarrow{d} N(0, \Sigma_C),$$

where

$$\Sigma_C = \left(\frac{\partial C(Y \underset{h}{\longrightarrow} X|Z)}{\partial(vec(\Pi)', vech(\Sigma_u)')}\right)\Omega\left(\frac{\partial C(Y \underset{h}{\longrightarrow} X|Z)}{\partial(vec(\Pi)', vech(\Sigma_u)')}\right)',$$

and

$$\Omega = \begin{pmatrix} \Gamma^{-1} \otimes \Sigma_u & 0 \\ 0 & 2(D_3' D_3)^{-1} D_3' (\Sigma_u \otimes \Sigma_u) D_3 (D_3' D_3)^{-1} \end{pmatrix}$$

where

vec denotes the column stacking operator,

$vech$ is the column stacking operator that stacks the elements on and below the diagonal only, and

D_3 is the duplication matrix defined such that $vech(F) = D_3 vech(F)$ for any symmetric 3×3 matrix F.

Remark 1 *It is not easy to calculate analytically the derivative of* $C(Y \underset{h}{\longrightarrow} X|Z)$.

Two ways to calculate the confidence intervals of $\hat{C}(Y \underset{h}{\longrightarrow} X|Z)$:

<u>*First one:*</u> use the simulation technique and calculate the derivative numerically.

<u>*Second one:*</u> calculate the bootstrap confidence interval.

Conditionally on sample we show the following result. Under assumptions 1 and 2, we have

$$(T-p)^{1/2}(\hat{C}^{*}(Y \underset{h}{\longrightarrow} X|Z) - \hat{C}(Y \underset{h}{\longrightarrow} X|Z)) \xrightarrow{d} N(0, \Sigma_C),$$

where $\hat{C}^{*}(Y \underset{h}{\longrightarrow} X|Z)$ represents the bootstrap estimator of our causality measure and Σ_C is the same as in the previous slide.

Remark 2 *If we suppose that the process* W *follows a* $VAR(\infty)$, *then we can use Inoue and Kilian's (2002) approach to get similar results.*

6 Empirical illustration

6.1 Objective:

To quantify the degree of relationships between macroeconomic and financial variables.

6.2 Data set:

Consists of monthly observations on nonborrowed reserves (NBR), the federal funds rate (r), the GDP deflator (P), and real GDP (GDP).

Our sample goes from January 1965 to December 1996 for a total of 384 observations.

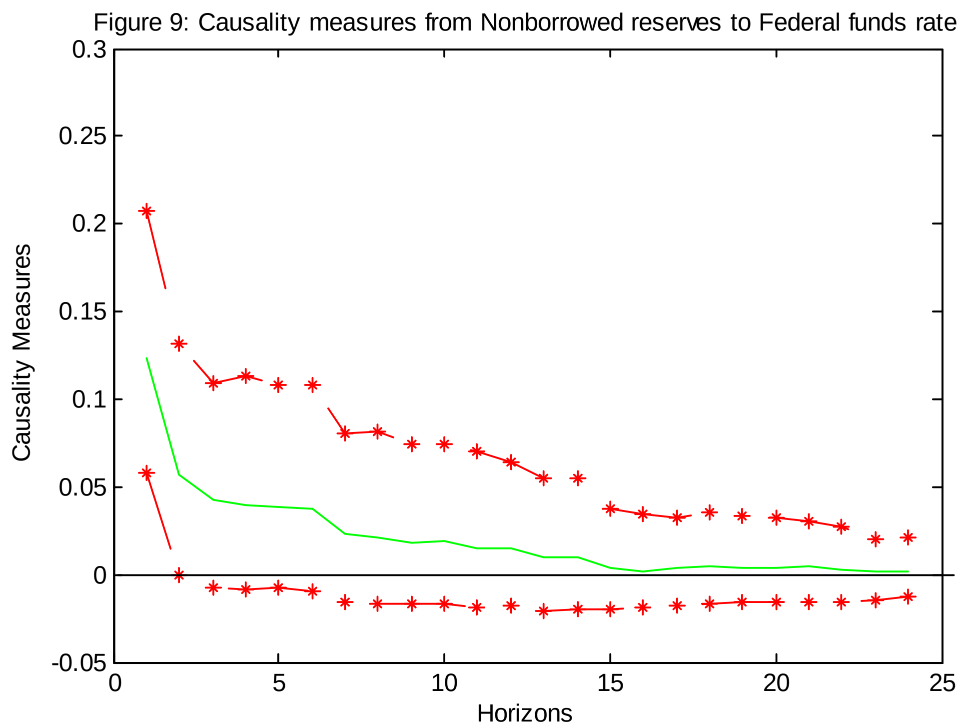
6.3 Calculation of causality measures:

We calculate different causality measures for horizons $h = 1, ..., 16$. Higher values of measures indicate greater causality.

We also calculate the corresponding nominal 95% bootstrap confidence intervals.

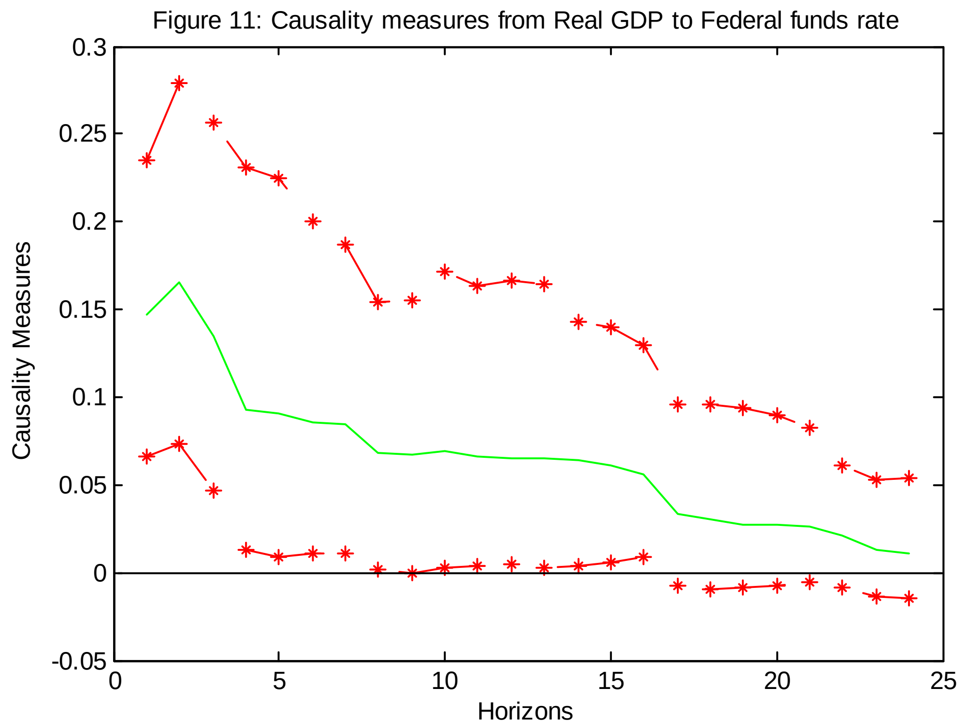
6.4 Some interesting results:

1. There is an important effect of nonborrowed reserves (NBR) on federal funds rate (r) at horizon 1 (one month ahead).



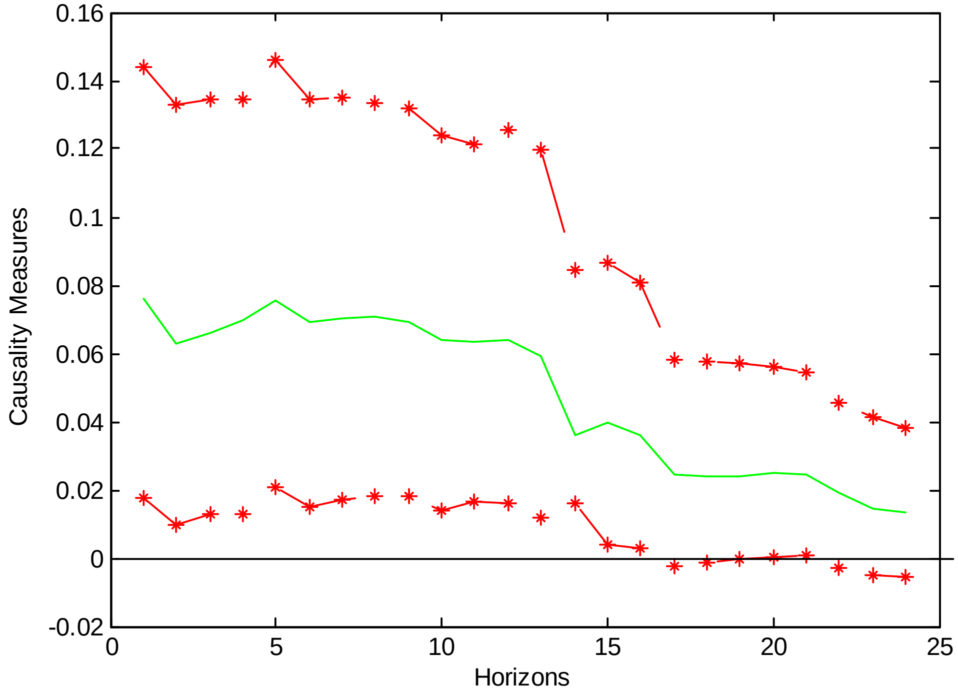
We can conclude that the nonborrowed reserves cause the federal funds rate only at short-term.

2. The causality from real GDP (GDP) to Federal funds rate (r) is very “important” at horizons 1, 2, and 3.



3. The Federal funds rate causes the real GDP until horizon 16.

Figure 13: Causality measures from Federal funds rate to Real GDP



7 Conclusion

7.1 We propose a parametric and nonparametric causality measures for any horizon $h \geq 1$. These measures are able to quantify the degree of an indirect effect.

7.2 The parametric measures are defined in terms of impulse response coefficients of $VMA(\infty)$ representation.

7.3 We propose a simple technique based on simulation to calculate our causality measures.

7.4 We establish the asymptotic distribution and the asymptotic validity of the residual-based bootstrap of the causality measure estimate.

7.5 From an empirical application we find that

* Nonborrowed reserves causes the federal funds rate only at

short-term.

* The causality from real GDP (GDP) to Federal funds rate (r) is very “important” at horizons 1, 2, and 3.

* The Federal funds rate causes the real GDP until horizon 16.