Spectral Methods and Finite Elements: APMA 935 2019

Course description: This course is a mathematical introduction to spectral methods and the finite element method. The emphasis will be on both the analysis and the implementation of these methods.

At the heart of both these methods is the same idea— the approximation of the solution by a (truncated) series expansion in terms of trial functions. Spectral methods use basis functions which are infinitely differentiable global functions, while finite element methods use localized trial functions. We introduce these ideas systematically. We examine the approximation properties of popular methods, and will learn how to choose and design the appropriate method for a given problem.

Topics covered: We will begin by solving PDEs using Chebfun for spectral methods (within MATLAB) and FreeFem++/FeNics and deal.ii for FEM, treating these packages as black-boxes. This will be a very fast-paced section.

With some computational experience in hand, we will examine the Galerkin and collocation spectral methods (the former are easier to analyse, the latter are easier to use) for ordinary and partial differential equations. We shall then look at the finite element method for elliptic and parabolic pde. The theoretical analysis of these methods will be the major intellectual component of the course. For both sets of methods, we will return to our initial coding experiments to study issues of stability, accuracy and convergence.

Applications include eigenvalue problems, Poisson’s problem, Schrödinger’s equation, assorted boundary-value problems, and wave equations. We’ll look at pattern formation on the sphere, and also study problems in gas dynamics and fluid mechanics (including the flow of a compressible fluid in a heating domain). The solution of Poisson problems on the square is neither our final goal nor our intermediate interest.

Prerequisites: A good (advanced undergraduate/beginning graduate) background in linear algebra, partial differential equations and numerical analysis/scientific computing. Coding will be done on a variety of platforms, so you gain familiarity with high-level packages for spectral and finite elements.

Assessment: There will be 5 assignments involving both computation and theoretical work, through the term, for a total of 75% of the grade. In addition, students will work on a term project worth 25% of the grade.

References: There are no prescribed textbooks for this course, though we will draw on material from many texts:

- Numerical Solutions of PDE by the FEM by Claes Johnson,
- The mathematical theory of FEM by Brenner and Scott,
- Spectral methods in Matlab by L.N. Trefethen,
- Spectral methods in fluid dynamics by Canuto, Hussaini, Quarteroni and Zang,
- Numerical approximations of PDE by Quarteroni and Valli