Key for 2006 Practice Exam #1

Note: There is no attempt here to describe all possible correct answers. In many cases other approaches to a question could garner full marks.
For the examiners, apart from the accuracy of the answers, the crucial test is whether the student has made clear the principles and/or method being used and whether those principles and/or method are sound.

1. For each of the following evaluate the limit if it exists and explain why it does not otherwise.

[2] (a) \( \lim_{x \to 1} \frac{x^3 - 1}{x - 1} \)

\( \text{ANSWER: } 3 \)

\text{JUSTIFICATION}

For \( x \neq 1 \), \( \frac{x^3 - 1}{x - 1} = x^2 + x + 1 \). Thus the given limit is the same as

\( \lim_{x \to 1} (x^2 + x + 1) = \left( \lim_{x \to 1} x \right)^2 + \left( \lim_{x \to 1} x \right) + \lim_{x \to 1} 1 = 1 + 1 + 1 = 3. \)

[3] (b) \( \lim_{x \to 0} \frac{|x|}{x} \)

\( \text{ANSWER: } \text{does not exist} \)

\text{JUSTIFY YOUR ANSWER}

For \( x < 0 \), \( |x|/x = -x/x = -1 \). Thus \( \lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} -1 = -1. \)

Similarly, \( \lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} 1 = 1 \). Since the one-sided limits are different, the given limit does not exist.

[3] (c) \( \lim_{x \to 1^-} \frac{1 - x}{(\pi/2) - \sin^{-1} x} \)

\( \text{ANSWER: } 0 \)

\text{JUSTIFY YOUR ANSWER}

By L'Hospital's Rule the given limit is equal to \( \lim_{x \to 1^-} \frac{-1}{-1/\sqrt{1 - x^2}} \) provided the latter exists. But the latter limit is equal 0 since \( \sqrt{1 - x^2} \to 0 \) as \( x \to 1^- \).
2. Let \( f(x) = \frac{\sec x - \tan x}{\sec x + \tan x} \).

(a) Find an expression for \( f'(x) \)

\[
\frac{-2 \sec x (\sec x - \tan x)}{\sec x + \tan x}
\]

(b) Simplify \( f'(x) / f(x) \)

\[
-2 \sec x
\]

3. Let \( l_t \) denote the tangent line to the parabola \( y = -x^2 \) at the point \( (t, -t^2) \).

(a) Find the equation of \( l_t \).

\[
y + 2tx = t^2
\]

EXPLANATION

The slope of \( l_t \) is \( \frac{dy}{dx}_{x=t} = -2t \). Thus the point slope equation of \( l_t \) is \( y - (-t^2) = (-2t)(x - t) \).
[2] (b) Assuming that \( l \) meets the hyperbola \( xy = 1 \) in two points find the \( x \)-coordinates of those points.

**ANSWER:**

\[
\frac{t^2 \pm \sqrt{t^4 - 8t}}{4t}.
\]

**EXPLANATION**

Substituting \( 1/x \) for \( y \) in the equation for \( l \) we get \((2t)x^2 + (-t^2)x + 1 = 0\). The roots are

\[
\frac{t^2 \pm \sqrt{t^4 - 8t}}{4t}.
\]

[2] (c) Find a value of \( t \) such that the tangent line \( l \) is also tangent to \( xy = 1 \).

**ANSWER:**

\( t = 2 \)

**EXPLANATION**

The slope of the tangent line is negative at every point of \( xy = 1 \). Therefore, if \( l \) is tangent to \( xy = 1 \), then \( t > 0 \). Also, by inspection of the roots from (b), when \( t > 0 \), the point(s) of intersection of \( l \) with the hyperbola lie in the first quadrant. The natural place to look for a tangent is where the two points of intersection coincide, i.e., where \( t^4 - 8t = 0 \). Since \( t > 0 \), this gives \( t = 2 \). The corresponding point of \( xy = 1 \) is \((1/2, 2)\). Since the slope of \( xy = 1 \) at \((1/2, 2)\) is \(-4\), \( l_2 \) is indeed tangent to \( xy = 1 \).

[6] 4. Consider the curve whose equation is \( 2(x^2 + y^2)^2 = 25xy \).

Find all points of the curve at which \( \frac{dy}{dx} = 0 \).

**ANSWER:**

\[
\left( \frac{5 \cdot 3^{1/4}}{4\sqrt{2}}, \frac{5 \cdot 3^{3/4}}{4\sqrt{2}} \right), \left( -\frac{5 \cdot 3^{1/4}}{4\sqrt{2}}, -\frac{5 \cdot 3^{3/4}}{4\sqrt{2}} \right)
\]

**SHOW YOUR WORK**

Implicit differentiation gives \( 4(x^2 + y^2) \left( 2x + 2\frac{dy}{dx} \right) = 25 \left( y + x \frac{dy}{dx} \right) \). Setting \( \frac{dy}{dx} = 0 \) we get \( x^2 + y^2 = 25y/8x \). Substituting for \( x^2 + y^2 \) in the original equation, we get \( 2 \left( 25y/8x \right)^2 = 25xy \) which gives \( 25y = 32x^3 \). Substituting for \( 25y \) now gives \( (x^2 + y^2)^2 = 16x^4 \), which gives \( y/x = \pm \sqrt{3} \). Clearly, \( y/x > 0 \). So \( y = \sqrt{3}x \). The rest is straightforward.
5. (a) Find the general antiderivative of \( \frac{1}{1 + 4x^2} \).

**ANSWER:**

\[
\frac{1}{2} \arctan(2x) + C
\]

**SHOW YOUR WORK**

The starting point is \( \frac{d}{dx} \arctan x = \frac{1}{1 + x^2} \). By the Chain Rule, \( \frac{d}{dx} \arctan 2x = \frac{2}{1 + (2x)^2} \).

(b) Find a function \( y \) defined on \((0, \infty)\) such that

\[
\frac{dy}{dx} = 2x + \frac{1}{x^2}, \quad y(1) = 0.
\]

**ANSWER:**

\[
x^2 - \frac{1}{x}
\]

**SHOW YOUR WORK**

Taking antiderivatives we have

\[
y = \int 2x \, dx + \int \frac{1}{x^2} \, dx = x^2 - \frac{1}{x} + C.
\]

To get \( y(1) = 0 \) we need \( C = 0 \).
6. A spherical planet has radius $R$ and a narrow hole bored along a diameter. An object falling towards the centre through the hole satisfies the differential equation

$$\frac{d^2x}{dt^2} = -c^2x$$

where $c$ is constant, $x$ the distance from the centre, and $t$ the elapsed time.

Initially, i.e. at $t = 0$, the object is at rest at the surface of the planet.

The motion of the falling object is given by an equation

$$x = A \cos ct + B \sin ct.$$ 

Given that $R = 2 \times 10^7$ and $c = 4 \times 10^{7/2}$, compute the values of the constants $A, B$ from the initial data.

Show that the time taken for the object to fall to the centre is $\pi/2c$.

EXPLANATION

Differentiating with respect to $t$ we get

$$\frac{dx}{dt} = -Ac \sin ct + Bc \cos ct.$$ 

Since $(x)_{t=0} = R$ and $\left( \frac{dx}{dt} \right)_{t=0} = 0$, we have $A = R$ and $B = 0$. Thus $x = R \cos ct$.

The least $t > 0$ for which $x = 0$ is given by $ct = \pi/2$, which is the same as $t = \pi/2c$.

7. A rectangle $ABCD$ with sides of length $x, y$ is inscribed in a semicircle of radius 1 as shown in the figure.

The rectangle is chosen so that its perimeter $2x + 2y$ is as large as possible.

Show that $xy = 4/5$.

EXPLANATION

Let $P = 2x + 2y$. Then $dP/dx = 2 + 2y/dx$. From Pythagoras’ theorem we have $x^2 + (y/2)^2 = 1$, which is $4x^2 + y^2 = 4$. Differentiating we get $8x + 2y dy/dx = 0$, so $dP/dx = 2 - (8x/y)$. Setting $dP/dx = 0$ we get $y = 4x$. Together with $4x^2 + y^2 = 4$ this gives $x = 1/\sqrt{5}$ and $y = 4/\sqrt{5}$. Hence $xy = 4/5$. As $x$ increases through $1/\sqrt{5}$, $dP/dx$ changes from negative to positive. So the critical point gives a maximum for $P$. 

\[\text{ANSWER:}\] 

\[A = R, \ B = 0\]
8. Let \( f(x) = \sqrt[4]{10^{100}} + x - 10^{25} \).

(a) Give the best possible linear estimate for the function \( f(x) \) for small values of \( x \), i.e., for \( x \) close to 0.

A linear estimate is a function \( Ax + B \), where \( A \) and \( B \) are constants to be determined.

ANSWER: 
\[(10^{-75}/4) x\]

SHOW YOUR WORK

Note that \( f(0) = 0 \), so \( B = 0 \). Also,

\[
\lim_{x \to 0} \frac{f(x)}{x} = f'(0) = \left(\frac{1}{4}\right) \left(x + 10^{100}\right)^{-3/4} \bigg|_{x=0} = 10^{-75}/4.
\]

So \( A = 10^{-75}/4 \).

(b) Is your linear estimate an underestimate (meaning \( Ax + B \leq f(x) \) whenever \( |x| < 1 \)), an overestimate (meaning \( Ax + B \geq f(x) \) whenever \( |x| < 1 \)), or neither?

Explain your answer using the Mean Value Theorem or any other theorem known to you which fits the context.

ANSWER

Let \( \alpha \) denote \( 10^{-75}/4 \). The key point is that \( f'(x) \) is decreasing for \( x > -10^{100} \).

Consider \( x > 0 \). By the MVT there exists \( c, \ 0 < c < x \), such that \( f(x) = xf'(c) \). So \( f(x) < xf'(0) = \alpha x \).

Now consider \( x \) such that \(-1 < x < 0 \). By the MVT there exists \( c, \ x < c < 0 \), such that \( f(x) = xf'(c) \). So again \( f(x) < xf'(0) = \alpha x \). We conclude that \( \alpha x \) is an overestimate of \( f(x) \). 

9. Newton’s law of cooling may be stated as:

\[
\frac{dT}{dt} = kT,
\]

where \( T \) is the difference between the temperature of a specified object at time \( t \) and the ambient temperature (usually a constant), \( t \) is the elapsed time, and \( k \) is a constant which depends on the particular case.

A kettle initially at 100\(^\circ\) (Celsius) cools to 80\(^\circ\) in one minute. The ambient temperature is constant at 20\(^\circ\).

Show that the time taken for the kettle to cool from 80\(^\circ\) to 60\(^\circ\) is:

\[
\frac{\ln(2/3)}{\ln(3/4)} \approx 1.4 \text{ minutes}
\]

**EXPLANATION**

The differential equation is the classical decay equation. It can be rewritten as \( d(ln T)/dt = k \). Taking antiderivatives we get \( \ln T = kt + C \). Let \( u \) denote the time from 80\(^\circ\) to 60\(^\circ\) in minutes. Substituting the data for \( t = 0 \), \( t = 1 \), and \( t = 1 + u \), we get

\[
\ln 80 = C, \quad \ln 60 = k + C, \quad \ln 40 = k(1 + u) + C.
\]

Subtracting the first equation from the second, and the second from the third, gives \( \ln(3/4) = k \) and \( \ln(2/3) = ku \). Hence the result.

10. Let \( f(x) = x^3(x - 1)^2 \).

(a) Find the largest interval on which \( f(x) \) is decreasing.

**ANSWER:**

**SHOW YOUR WORK**

Note that \( f(x) = x^5 - 2x^4 + x^3 \). Therefore

\[
f'(x) = 5x^4 - 8x^3 + 3x^2 = x^2(5x - 3)(x - 1).
\]

By inspection, \( f'(x) \geq 0 \) for all \( x \leq 3/5 \) and all \( x \geq 1 \), while \( f'(x) < 0 \) on \((3/5, 1)\). So \([3/5, 1]\) is the largest interval on which \( f(x) \) is decreasing.
(b) Find all points of inflection of the function \( f(x) \).

**ANSWER:** 
\[ 0, \frac{6 \pm \sqrt{6}}{10} \]

**SHOW YOUR WORK**

Continuing our work above, we see that

\[
 f''(x) = 20x^3 - 24x^2 + 6x = x \left( x - \frac{6 + \sqrt{6}}{10} \right) \left( x - \frac{6 - \sqrt{6}}{10} \right).
\]

Note that \( f''(x) \) changes sign at each of its three zeros. So each zero of \( f''(x) \) is an inflection point of \( f(x) \).

---

11. At time \( t = 0 \) plane \( A \) is traveling NW at 500 mph while plane \( B \), 10 miles north of plane \( A \), is traveling NE at 500 mph.

The planes are travelling with constant speed and direction.

Let \( f(t) \) denote the distance between the planes so that \( f(0) = 10 \).

Show that \( f'(0) = 0 \).

**SHOW YOUR WORK**

As origin \( O \) of coordinates take the initial position of plane \( A \). Let the positive \( x \)-axis point west and the positive \( y \)-axis point north. Then after \( t \) hours the respective positions of planes \( A, B \) are

\[
(250\sqrt{2}t, 250\sqrt{2}t), \quad (-250\sqrt{2}t, 10 + 250\sqrt{2}t).
\]

So

\[
f(t)^2 = (2 \cdot 500^2)t^2 + 10^2.
\]

Differentiating we get

\[
2f(t)f'(t) = (4 \cdot 500^2)t.
\]

Letting \( t = 0 \) we get \( 2f(0)f'(0) = 0 \). Therefore \( f'(0) = 0 \).
[3] 12. (a) Let \( f(x) \) be a function whose domain includes \((0, \infty)\) and \( c > 0 \). Define \( f'(c) \) as a limit.

**DEFINITION**

\[
f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}
\]

[5] (b) Using only the limit laws and the definition given in part (a) show that, if \( f(x) = 1/x \), then

\[ f'(c) = -1/c^2 \quad (c > 0). \]

**EXPLANATION**

Observe that, when \( x \neq c \),

\[
\frac{f(x) - f(c)}{x - c} = \frac{(1/x) - (1/c)}{x - c} = \frac{(c - x)/(xc)}{x - c} = -\frac{1}{xc}.
\]

(2)

Suppose \( c > 0 \). Then, from (2),

\[
f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} -\frac{1}{xc} = -\frac{1}{c} \lim_{x \to c} \frac{1}{x} = -\frac{1}{c^2} \left( 1 / \lim_{x \to c} x \right) = -\frac{1}{c^2}.
\]

This is enough.
13. A water tank is in the shape of an inverted right circular cone. It has diameter 6 feet at the top and height 4 feet. The tank is being filled with water at a rate of 10 cubic feet per minute. A leaf is floating in the centre of the tank and moving along the axis of the tank as the water level rises.

(a) What is the velocity of the leaf when the water level is 2 feet?

**ANSWER:** \( \frac{40}{9\pi} \) feet/minute

**SHOW YOUR WORK**

Let \( V(x) \) denote the volume of water in the tank when the depth is \( x \). By the formula for the volume of a cone, \( V(4) = \frac{\pi}{3} \cdot 4 \cdot 3^2 = 12\pi \) cubic feet. When the depth is \( x \) all the linear dimensions of the cone are reduced in the proportion \( x : 4 \). Therefore

\[
V(x) = 12\pi \left( \frac{x}{4} \right)^3 = \frac{3\pi}{16} x^3.
\]

Differentiating with respect to time \( t \), measured in minutes,

\[
10 = \frac{dV}{dt} = \frac{9\pi}{16} x^2 \frac{dx}{dt}.
\]

Letting \( x = 2 \) we get \( \frac{dx}{dt} = \frac{40}{9\pi} \).

(b) What is the acceleration of the leaf when the water level is 2 feet?

**ANSWER:** \(-\left( \frac{40}{9\pi} \right)^2\) feet/minute

**SHOW YOUR WORK**

Differentiating (3) we get

\[
0 = 2x \left( \frac{dx}{dt} \right)^2 + x^2 \frac{d^2x}{dt^2}.
\]

Substituting \( x = 2 \), we get

\[
\frac{d^2x}{dt^2} \bigg|_{x=2} = -\frac{2}{x} \left( \frac{dx}{dt} \right)^2 \bigg|_{x=2} = -\left( \frac{40}{9\pi} \right)^2.
\]
[6] 14. Compute the area enclosed between the parabolas $y = 1 - x^2$ and $y = x^2 - 1$. Note that these parabolas are mirror images of each other in the $x$-axis.

**ANSWER:** $\frac{8}{3}$

SHOW YOUR WORK

Note that the parabolas intersect at $(-1, 0)$ and $(1, 0)$. The area between them is bounded above by $y = 1 - x^2$ ($x \in [-1, 1]$), and below by $y = x^2 - 1$ ($x \in [-1, 1]$).

We need the principle that, if $F(x)$ is an antiderivative of $f(x)$, then the area under $y = f(x)$ between $x = a$ and $x = b$ is $F(b) - F(a)$. By symmetry the area between the parabolas is twice the area below $y = 1 - x^2$ between $x = -1$ and $x = 1$. By inspection, $F(x) = x - \frac{1}{3}x^3$ is an antiderivative of $1 - x^2$. So the desired area is

$$2 \left( F(1) - F(-1) \right) = 2 \left[ \left( 1 - \frac{1}{3} \right) - \left( -1 - \frac{1}{3} \right) \right] = 8/3.$$