On the Joint Determination of Fiscal and Monetary Policy

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Abstract

In the absence of government commitment, the conduct of fiscal and monetary policy depends on the sign of inherited net nominal government obligations. When these obligations are negative, monetary policy is non-distortionary and fiscal policy distortions are smoothed over time, either forever or for a finite number of periods, depending on the initial state. For positive net nominal government obligations, both fiscal and monetary policies are distortionary, and there exists a unique and stable steady state. At this steady state, a reform endowing the government with a commitment technology has no effect on policy. For any level of initial debt, the estimated welfare loss due to lack of government commitment is small.

Keywords: money; inflation; government debt; time-consistency; lack of commitment.

JEL classification: E13, E52, E62, E63

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1 Introduction

Monetary policy and government debt are inextricably linked. First, the level of debt affects monetary policy decisions, since inflation can be used to reduce the real value of inherited nominal debt. Second, monetary policy affects fiscal policy decisions, since it determines how current deficits are going to be financed tomorrow, i.e., how future distortions are internalized today. Both these channels are theoretically and empirically relevant.\(^1\)

Following the tradition of Kiyotaki and Wright (1989, 1991, 1993), Lagos and Wright (2005) propose a tractable framework for the study of monetary policy, with explicit micro foundations for the role of fiat money.\(^2\) The literature spawned by Lagos and Wright (2005) has revisited classic questions in monetary economics\(^3\), but so far—with the exception of Aruoba and Chugh (2008)—has restricted attention to exogenous, stationary (i.e., constant) government policy. Two questions arise naturally. First, is the Lagos-Wright framework with the addition of endogenous government policy suitable for positive analysis, i.e., are its predictions compatible with observations on actual economies? Second, what are the model’s implications for the determinants of policy and the design of institutions?

To answer these questions, I consider a variant of Lagos and Wright (2005), with the addition of a benevolent government that cannot commit to future policy choices and uses money, nominal bonds and (distortionary) taxes to finance the provision of a valued public good. As shown in Martin (2009), government lack of commitment links monetary policy and debt as described above, and provides a mechanism that explains the level of debt.\(^4\)

How the level of debt affects monetary policy depends on the sign of inherited net nominal government obligations (i.e., money plus bonds). When net obligations are positive, there is an incentive to inflate above the Friedman rule to reduce the financial burden; these incentives increase with the debt-money ratio. Conversely, when net obligations are negative, the incentives go in the opposite direction and monetary policy is thus, non-distortionary, i.e., the marginal values of money and bonds are equalized.\(^5\) Hence, monetary policy exhibits a “kink” at zero nominal obligations.

If nominal claims on the private sector (i.e., negative net government obligations) are sufficiently large, then the government perfectly smooths distortions across periods, which determines a continuum of steady states, including the first-best. For this range of nominal claims, the government behaves as in Barro (1979). If the inherited claims are positive but insufficient to implement perfect distortion-smoothing, then distortions are smoothed for only a finite number of periods by rolling-off debt. Due to the “kink” in monetary policy described above, policy in this range features

1Ohanian (1998) provides a thorough historical account for the U.S. economy. Sargent and Wallace (1981) first showed how the effects of monetary policy are affected by a given fiscal policy. See Martin (2009) for a related discussion.

2Shi (1997) provides an alternative framework, with similar virtues.

3See Aruoba and Wright (2003), Rocheteau and Wright (2005), Lagos and Rocheteau (2005, 2008), Berentsen, Camera and Waller (2007), among many others

4There are alternative mechanisms. Battaglini and Coate (2008) show that inefficiencies due to pork-barrel spending provide an explanation for the distribution of (real) debt in the long-run. Diamond (1965), Aiyagari and McGrattan (1998) and Shin (2006) provide a role for debt by using it to reduce some dynamic inefficiency.

5In equilibrium, distortions from monetary policy cannot be negative. This bound on policy is typical of monetary economies, since no-arbitrage implies that in any monetary equilibrium the marginal value of money cannot exceed the marginal value of nominal bonds.
discontinuities at an infinite but countable number of critical debt levels.

When net nominal government obligations are positive, both fiscal and monetary policies are distortionary, and there exists a unique, stable steady state. Thus, for positive analysis one can restrict attention to positive net nominal government obligations. Under a set of conditions, long-run debt is shown to be larger than zero, as typically observed in actual economies.

Government policy depends critically on the curvature of consumption and the measure of buyers in anonymous (i.e., cash-based) markets, since both features affect how the government internalizes the cost of inflation and by extension, how costly it is to reduce the real value of inherited nominal debt. The first channel is also found in standard cash-in-advance models,\(^6\) whereas the second arises here since the model explicitly features a double coincidence of wants problem.

On the normative side, there are two main results. First, if the economy is at the steady state with positive net nominal obligations, then a reform that endows the government with a commitment technology has no effect on policy. This result arises despite the fact that both fiscal and monetary policies are distortionary in steady state, which contrasts with previous findings that link time-consistency with optimality of the Friedman rule (see Alvarez, Kehoe and Neumeyer, 2004). For debt levels outside the steady state, the welfare loss due to lack of commitment is estimated to be quite low. For example, for debt levels between zero and twice the post-war U.S. average, the welfare cost in a calibrated economy is at most equivalent to a one-time fee of 0.04\% of a year’s consumption.

Second, assuming parameters are such that steady state debt is positive, governments choose optimally to issue illiquid bonds, i.e., bonds that cannot be used as means of payment in some markets. This result presumes that the government is unable to change the liquidity properties of inherited debt. Furthermore, if society could commit to either liquid or illiquid government bonds, then starting at zero debt, it would choose to have illiquid bonds or equivalently impose a cash-in-advance constraint on some goods. In contrast to Kocherlakota (2003), illiquid bonds are essential here, not because they allow agents to intertemporally exchange money, but because they allow the government to trade distortions across periods.

In a closely related paper, Aruoba and Chugh (2008) formulate the Ramsey problem in the Lagos-Wright environment and provide some important new insights on optimal policy. In terms of the questions posed here, however, their approach is subject to the usual criticism of the Ramsey framework: the policy prescription is generically time-inconsistent and the model offers no meaningful prediction for the long-run level of debt (and thus, policy in general).

The paper is organized as follows. Section 2 presents the environment. Section 3 derives the monetary equilibrium given government policy. Section 4 formulates the government’s problem, characterizes equilibrium policy and derives the main theoretical results. Section 5 provides analytical and numerical analyses using suitable functional forms. Section 6 concludes.

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2 Environment

The environment is a variant of Lagos and Wright (2005). There is a continuum of infinitely lived agents. Each period, two perfectly competitive markets open in sequence: a day and a night market. In each stage a perishable good is produced and consumed. Before each day market opens, agents receive an idiosyncratic shock that determines whether they can produce or consume the day-good, \( x \). With probability \( \eta \in (0, 1) \) an agent wants to consume but cannot produce, while with probability \( 1 - \eta \) an agent can produce but does not want consume. A consumer derives utility \( u(x) \), where \( u \) is twice continuously differentiable, with \( u_x > 0 \) and \( u_{xx} < 0 \), and satisfies Inada conditions. A producer incurs in utility cost \( f(x) \), where \( f \) is twice continuously differentiable with \( f_x > 0 \) and \( f_{xx} \geq 0 \). Given the assumptions on \( u \) and \( f \) there exists a unique \( \hat{x} \in (0, \infty) \) such that \( u_x(\hat{x}) = f_x(\eta \hat{x} - (\frac{\eta}{1-\eta})^2) \).

Assume agents lack commitment and are anonymous, in the sense that private trading histories are unobservable. Thus, no private credit is possible. Since the day market features a lack of double coincidence of wants problem, some medium of exchange is essential for trade to occur.\(^7\)

At night, all agents can produce and consume the night-good, \( c \). The production technology is assumed to be linear in hours worked. Utility from consumption is given by \( U(c) \), where \( U \) is twice continuously differentiable, with \( U_c > 0 \), \( U_{cc} < 0 \), and satisfies Inada conditions. Disutility from labor is given by \( \alpha n \), where \( n \) is hours worked and \( \alpha > 0 \). Let \( \hat{c} \in (0, \infty) \) such that \( U_c(\hat{c}) = \alpha \).

There is a benevolent government that supplies a valued public good \( g \). To finance its expenditure, the government may use proportional labor taxes \( \tau \), print fiat money at rate \( \mu \) and issue one-period nominal bonds, which are redeemable in fiat money. The public good is transformed one-to-one from the night-good. Agents derive utility from the public good according to \( v(g) \), where \( v \) is twice continuously differentiable, with \( v_g > 0 \), \( v_{gg} < 0 \), and satisfies Inada conditions. Let \( \hat{g} \in (0, \infty) \) such that \( v_g(\hat{g}) = \alpha \).

The government can commit to policies within the period, but lacks the ability to commit to future policy choices. To characterize government policy with lack of commitment, I adopt the notion of Markov-perfect equilibrium, i.e., where policy is a function of fundamentals only.\(^8\)

Assume the government announces its policy for the period at the beginning of the day, before agents’ preference shocks are realized. The government only actively participates in the night-market, i.e., taxes are levied on hours worked at night and open market operations are conducted in the night market. As in Aruoba and Chugh (2008) and Berentsen and Waller (2008), public bonds are book-entries in the government’s record. Since bonds are not physical objects and the government does not participate in the day market (i.e., cannot intermediate or provide third-party verification), bonds are not used as a medium of exchange in the day market and thus, money is essential. Alternatively, we could assume that bonds—but not money—can be costlessly counterfeited.\(^9\) Again, bonds will only be traded in markets where the government participates. Government bonds are then illiquid as in Kocherlakota (2003). In section 4, I show that governments

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\(^{8}\)See Maskin and Tirole (2001) for a definition and justification of this solution concept. For applications to dynamic policy games see Klein, Kruell and Rios-Rull (2008), Díaz-Giménez, Giovannetti, Marimón and Teles (2008) and Martin (2009), among others.

\(^{9}\)There are several examples in the literature where fiat money is assumed to be the only recognizable asset in certain markets. E.g., see Aruoba, Waller and Wright (2007) and Telyukova and Wright (2008).
would optimally choose to issue illiquid bonds, under certain conditions consistent with actual economies.

All nominal variables—except for bond prices—are normalized by the aggregate money stock. Thus, today’s aggregate money supply is equal to 1 and tomorrow’s is $1 + \mu$. The government budget constraint is

$$1 + B + pg = p\tau n + (1 + \mu)(1 + qB'),$$

where $B$ is the current aggregate bond-money ratio, $p$ is the—normalized—market price of the night-good $c$, and $q$ is the price of a bond that earns one unit of fiat money in the following night market. “Primes” denote variables evaluated in the following period. Thus, $B'$ is tomorrow’s aggregate bond-money ratio.

### 3 Monetary Equilibrium

In this section, I derive the conditions that characterize a monetary equilibrium for a given government policy.

#### 3.1 The night market

An agent arrives to the night market with individual money balances $m$ and government bonds $b$. Since bonds are redeemed in fiat money at par, the composition of an agent’s nominal portfolio at the beginning of the night is irrelevant. Let $z \equiv m + b$, i.e., total—normalized—nominal holdings. The budget constraint of an agent at night is

$$pc + (1 + \mu)(m' + qb') = p(1 - \tau)n + z.$$

Notice that the composition of the nominal portfolio the agent decides to carry over to the next period matters, as only fiat money is used to buy goods in the day market.

Let $V(m, b)$ be the value of entering the day market with money balances $m$ and bond balances $b$, and let $W(z)$ be the value of entering the night market with total nominal balances $z$. After solving $n$ from (2), the problem of an agent in the night market is

$$W(z) = \max_{c, m', b'} U(c) - \alpha \frac{1}{1 - \tau} + \alpha \frac{(z - (1 + \mu)(m' + qb'))}{p(1 - \tau)} + v(g) + \beta V(m', b').$$

The first-order conditions are

$$U_c - \frac{\alpha}{1 - \tau} = 0$$

$$-\frac{\alpha(1 + \mu)}{p(1 - \tau)} + \beta V'_m = 0$$

$$-\frac{\alpha q(1 + \mu)}{p(1 - \tau)} + \beta V'_b = 0.$$
Assuming a symmetric equilibrium, we can follow Lagos and Wright (2005) to show that (4) and (5) imply all agents exit the night market with the same money and bond balances. Furthermore, the value function $W$ is linear, $W(z) = \frac{\alpha}{p(1-\tau)}$. Both results follow from labor entering linearly in the objective function and the budget constraint. The linearity of $W$ allows us to write $W(z) = W(0) + \frac{\alpha z}{p(1-\tau)}$.

From (4) and (5) we also get
\[ q = \frac{V'_c}{V'_m}. \] (6)
Thus, if $V'_b < V'_m$, i.e., if the value of entering tomorrow’s day market with a unit of bonds is less than with a unit of money, then agents need to be compensated to acquire bonds, i.e., $q < 1$.

### 3.2 The day market

The ex-ante value for an agent that enters the day market is $V(m, b) = \eta V^c(m, b) + (1-\eta)V^p(m, b)$, where $V^c$ and $V^p$ are the values of being a consumer and a producer in the day market, respectively.

A consumer faces a day-budget constraint, $\tilde{p}x \leq m$, where $\tilde{p}$ is the—normalized—market price of good $x$. Using $\xi$ as the Lagrange multiplier associated with this constraint, the problem of a consumer can be written as
\[ V^c(m, b) = \max_x u(x) + W(0) + \frac{\alpha(m + b - \tilde{p}x)}{p(1-\tau)} + \xi(m - \tilde{p}x). \]

The first-order condition is
\[ u_x - \frac{\alpha \tilde{p}}{p(1-\tau)} - \xi \tilde{p} = 0, \] (7)
which implies $\xi = \frac{u_x}{\tilde{p}} - \frac{\alpha}{p(1-\tau)}$. From the envelope condition we get $V^c_m = \frac{\alpha}{p(1-\tau)} + \xi = \frac{u_x}{\tilde{p}}$ and $V^c_b = \frac{\alpha}{p(1-\tau)}$. In terms of a consumer’s decision during the day, the amount of money brought into the period matters—the budget constraint is typically binding—whereas bond holdings do not.

In general, the individual quantities consumed and produced are different. Let $y$ be an individual producer’s output of the day-good. The problem of a producer is
\[ V^p(m, b) = \max_y -f(y) + W(0) + \frac{\alpha(m + b + \tilde{p}y)}{p(1-\tau)}. \]

The first-order condition is
\[ -f_y + \frac{\alpha \tilde{p}}{p(1-\tau)} = 0. \] (8)
Note that because of quasi-linear preferences in the night stage, a producer’s actions during the day are unaffected by the amount of money or bonds he brings into the period. The envelope condition

\footnote{One minor caveat is that $V$ is linear in $b$ and hence, a non-degenerate distribution of bonds is possible in equilibrium. Thus, the focus is on symmetric equilibria. Note that the agent’s day problem is unaffected by bond holdings (see below), while at night the agent only cares about total nominal holdings. See Aruoba and Chugh (2008) for a related discussion.}
implies $V_m^p = V_b^p = \frac{\alpha}{\rho(1-\tau)}$. We can now derive:

$$
V_m = \frac{\eta u_x}{\bar{p}} + \frac{(1-\eta)\alpha}{\rho(1-\tau)} \tag{9}
$$

$$
V_b = \frac{\alpha}{\rho(1-\tau)}. \tag{10}
$$

### 3.3 Equilibrium

The aggregate resource constraint in the day implies

$$
y = \frac{\eta x}{1-\eta}. \tag{11}
$$

A standard result is that consumers spend all their money in the day market, i.e., $m = \bar{p}x$. The market clearing condition is then $\eta = (1-\eta)\bar{p}y$, which implies $\bar{p} = \frac{1}{\eta}$. Substitute this expression in the first-order condition of the producer (8) and we get that $x$ solves

$$
f_yx = \frac{\alpha}{p(1-\tau)}, \tag{12}
$$

where $y$ satisfies (11). Then, from (9) and (10) we get $V_m = x(\eta u_x + (1-\eta)f_y)$ and $V_b = f_yx$. Equation (7) can be written as $\xi = x(u_x - f_y)$. In a monetary equilibrium $\xi \geq 0$, which implies $u_x - f_y \geq 0$, i.e., $V_m \geq V_b$. This inequality holds in every period and thus, $q \leq 1$.

As described above, all agents choose the same $c, m', b'$ in the night market. Thus, in equilibrium, $m' = 1$ and $b' = B'$. The only relevant difference between agents is their role in the day market and the corresponding labor effort in the night market. It is easy to verify that the aggregate resource constraint in the night market is satisfied, i.e.,

$$
c + g = n. \tag{13}
$$

We can now collect the remaining equations that summarize agents’ behavior in any given period. After some rearrangement we can write equations (3), (4), (6), (7) and (12) as

$$
\mu = \frac{\beta x'(\eta u_x' + (1-\eta)f_y')}{f_yx} - 1 \tag{14}
$$

$$
\tau = 1 - \frac{\alpha}{U_c} \tag{15}
$$

$$
p = \frac{U_c}{f_yx} \tag{16}
$$

$$
q = \frac{f_y'}{\eta u_x' + (1-\eta)f_y'} \tag{17}
$$

$$
{u_x - f_y} \geq 0, \tag{18}
$$

where $y, y'$ satisfy (11). As described above, (18) comes from the requirement that the Lagrange multiplier in the day-consumer’s problem, $\xi$, be non-negative.
4 Government Policy

In this section, I first formulate the problem of the government with lack of commitment and define 
Markov-perfect monetary equilibrium. Second, I characterize the theoretical properties of the 
equilibrium. Third, I analyze policy in the long-run. Fourth, I establish conditions under which 
iliquid bonds are optimal. Fifth, I explore the role of commitment.

4.1 Problem of the government

As mentioned above, the government is benevolent, but can only commit to within-period policy 
choices. Thus, in any given period, the problem of the current government is to maximize agents'
present value utility subject to its budget constraint. Note that the government budget constraint is 
a function of agents’ competitive behavior and future government policy. Use equilibrium conditions 
(13) to (17) to write the government budget constraint (1) as

\[(U_c - \alpha)c - \alpha g + \beta \eta x'(u' - f'_{y}) + \beta f'y'(1 + B') - f_yx(1 + B) = 0.\] (19)

The equation above is a function of \(B, B', x, x', c, g\). All these variables are chosen in 
the current period, except for \(x'\), which is a function of the policy implemented by the government 
tomorrow. In turn, future government policy is a function of the state variable, i.e., the bond-to-
money ratio. Thus, let \(x' = \mathcal{X}(B')\), where \(\mathcal{X}\) is the policy that the current government expects its 
future-self to follow. Given \(y = \eta x\) and \(y' = \eta \mathcal{X}(B')\), we can write the government budget constraint 
compactly as \(\varepsilon(B, B', x, \varepsilon(B'), c, g) = 0\). In the analysis that follows, we will make frequent use of the partial derivatives of this expression. Specifically, we have: \(\varepsilon_B = -f_yx; \varepsilon_{B'} = \beta f'y';\) 
\(\varepsilon_x = -(1 + B)(f_y + f_{yy}y); \varepsilon_{x'} = \beta \{\eta(u'_x + u''_{xx}x') + (1 - \eta + B')(f'_y + f''_{yy}y')\}; \varepsilon_c = U_c - \alpha + U_{cc}c\); and \(\varepsilon_g = -\alpha\).

The expression for \(\varepsilon_x\) shows how the level of debt affects monetary policy. If \((1 + B) > 0\) then 
the government would like to reduce \(x\) to relax its budget constraint. From (14), a lower \(x\) is 
implemented through a higher \(\mu\). In other words, if net nominal obligations are positive, then the 
government has an incentive to use inflation to reduce the real value of its financial burden. On the 
other hand, if \((1 + B) < 0\) then the government has the opposite incentive. In this case, (18) becomes 
a binding constraint, since we restrict the current government to implement policies consistent with 
a monetary equilibrium.

Given the perception that future governments implement \(\mathcal{X}(B)\), the problem of the current 
government is

\[\mathcal{V}(B) = \max_{B', x, c, g} \eta u(x) - (1 - \eta)f(y) + U(c) - \alpha(c + g) + v(g) + \beta \mathcal{V}(B')\]

subject to

\[\varepsilon(B, B', x, \mathcal{X}(B'), c, g) = 0\]
\[u_x - f_y \geq 0.\]
Definition 1 Let $\Gamma \equiv [B^L, B^H]$, where $-\infty < B^L < B^H < \infty$. A Markov-Perfect Monetary Equilibrium (MPME) is a set of functions $\{B, X, C, G, V\} : \mathbb{R} \to \mathbb{R}^5$, such that for all $B \in \Gamma$:

(i) $\{B(B), X(B), C(B), G(B)\} = \arg \max_{B' \in \Gamma, x, c, g} \eta u(x) - (1 - \eta)f(y) + U(c) - \alpha(c + g) + v(g) + \beta V(B')$

subject to $\varepsilon(B, B', x, X(B'), c, g) = 0, u_x - f_y \geq 0$, where $y = \frac{nx}{1 - \eta}$ and $y' = \frac{\eta X(B')}{1 - \eta}$; and

(ii) $V(B) = \eta u(X(B)) - (1 - \eta)f\left(\frac{\eta X(B)}{1 - \eta}\right) + U(C(B)) - \alpha(C(B) + G(B)) + v(G(B)) + \beta V(B(B))$.

The bounds on debt specified above are necessary for a proper definition of the equilibrium at this point, since equations (18) and (19) are not sufficient to characterize a monetary equilibrium, which is a standard feature in this type of problem. Specifically, (18) and (19) do not rule out the possibility of the government running a Ponzi-scheme. Due to the lack-of-commitment friction, there is no single government on which we can impose an additional present-value “pay-back” (i.e., no Ponzi games) constraint. I will address this issue below by showing a more general property of the MPME, which in turn rules-out Ponzi schemes. The characterization of the equilibrium that follows will not rely on specific bounds on debt.

4.2 Equilibrium characterization

The problem of the government is not a standard dynamic programming problem, since the current government takes $X(B)$, i.e., the policy of future governments, as given. Solving for a MPME involves finding the fixed point of both $V(B)$ and $X(B)$. If these two functions are differentiable (a.e.), we can use the first-order condition to further characterize the MPME. Typically, one would impose differentiability of the policy functions as a refinement on the Markov-perfect equilibrium (more on this below). In this case however, since the non-negativity constraint (18) will bind for some debt-levels, the restriction needs to be weaker.

Assumption 1 The current government expects future governments to implement policy $X(B)$ which is continuous and differentiable almost everywhere.

The assumption above rules out equilibria where policy functions are non-differentiable for non-fundamental reasons. Specifically, if $X(B)$ is expected to be discontinuous, the discontinuity may be preserved in the policy response of the current government, since small changes in debt will sometimes trigger big changes in future policy. However, the expected discontinuity would not be rooted in environment fundamentals. These type of non-differentiable equilibria are an artifact of the infinite horizon, as they would typically not exist in versions of the economy with a finite horizon (and appropriate terminal conditions for the value of money).11 Note that Assumption 1 does not rule out the possibility that other equilibrium policy functions are non-differentiable. In fact, as shown below, $B(B), C(B)$ and $G(B)$ will all show discontinuities for certain debt levels.

11See Krusell and Smith (2003) and Martin (2009) for further characterization and discussion. Krusell, Martin and Rios-Rull (2006) analyze an economy in which the fundamental equilibrium is non-differentiable due to an endogenous upper bound on debt.
However, these discontinuities will be rooted in fundamentals; specifically, a “kink” in monetary policy due to (18).

Using $\lambda$ and $\zeta$ as the Lagrange multipliers associated with the constraints of the government’s problem, the first-order conditions are

$$
\lambda(\varepsilon_B' + \varepsilon_x'X_B') + \beta\lambda'\varepsilon_B = 0 \quad (20)
$$

$$
\eta(u_x - f_y) + \lambda\varepsilon_x + \zeta\left(u_{xx} - \frac{\eta f_{yy}}{1-\eta}\right) = 0 \quad (21)
$$

$$
U_c - \alpha + \lambda\varepsilon_c = 0 \quad (22)
$$

$$
-\alpha + v_g + \lambda\varepsilon_g = 0 \quad (23)
$$

where we use the envelope condition, $V_B = \lambda\varepsilon_B$, to simplify (20).

Notice that (20) has the derivative of the unknown function $X'$, evaluated at $B'$. For this reason, this equation is usually referred to as a Generalized Euler Equation (GEE). The presence of $X_B'$ reflects the time-consistency problem: the current government takes into account how its actions affect tomorrow’s government decisions, whereas the government tomorrow does not internalize how its policy affected past decisions.

Let us now take a closer look at the dynamic incentives faced by the government. Notice that $\varepsilon_B' = -\beta\varepsilon_B' = \beta f_g x'$ and thus, (20) can be written as

$$
\beta f_g x' (\lambda - \lambda') + \lambda\varepsilon_x'X_B' = 0, \quad (24)
$$

where, from (22) and (23), it follows that $\lambda$ is a direct measure of the distortions created by current policy. Since $\varepsilon_B' = -\beta\varepsilon_B'$, the governments assign equal weight to distortions today and tomorrow. Thus, the term $\beta f_g x' (\lambda - \lambda')$ in (24) is the standard trade-off between these distortions. Absent any other margins, this term implies the government would perfectly smooth distortions over time, i.e., set $\lambda = \lambda'$. This is what the debt model by Barro (1979) would imply in the absence of shocks, and is also a typical feature of models with government commitment, where policy is constant after the initial period (see section 4.5 below). However, as explained above, with lack of commitment, the term $\lambda\varepsilon_x'X_B'$ in (24) appears. In this case, how the current government substitutes distortions across periods depends crucially on how future policy reacts to different levels of inherited debt. This is the channel through which monetary policy affects debt policy.

The following auxiliary result will be useful in the analysis of the equilibrium. For clarity of exposition, all proofs are in the Appendix.

**Lemma 1** In a MPME, $\lambda = 0$ if and only if $\{\hat{x}, \hat{c}, \hat{g}\}$ is implemented in the current and all future periods.

In other words, $\lambda = 0$ if and only if the first-best is implemented. The Lemma implies that a government will not implement the first-best allocation $\{\hat{x}, \hat{c}, \hat{g}\}$ in the current period if there are distortions in the future. From (19), the steady state debt level that implements the first-best is

$$
\hat{B} = -1 - \frac{\alpha \hat{g}}{(1-\beta)\hat{u}_x \hat{x}}.
$$
Examining the problem of the government and the first-order conditions, it seems plausible that there exists a MPME featuring \( \mathcal{X}(\lambda) = \hat{x} \) for all \( \lambda \). Suppose this is the case. Hence, \( \mathcal{X}_B = 0 \) for all \( B \) and thus, from (24), \( \lambda = \mathcal{X} \). Then, (22) and (23) imply \( c = c' \) and \( g = g' \). Note that \( \varepsilon_x < 0 \) for \( B > -1 \), which from (21) implies \( \lambda = \zeta = 0 \) for all \( B > -1 \). Given this policy, it is also feasible to implement \( \{\hat{x}, \hat{c}, \hat{g}\} \) for \( B \leq -1 \). Thus, our candidate equilibrium is \( \{\mathcal{X}(\lambda) = \hat{x}, \mathcal{C}(B) = \hat{c}, \mathcal{G}(B) = \hat{g}\} \), where from (19), \( \mathcal{B}(B) = \frac{\alpha g}{\beta u_x x} + \frac{1+B}{\beta} - 1 \). In other words, the government implements the first-best in every period through a Ponzi-scheme of ever-increasing debt. The proposition below shows that this policy cannot be an equilibrium, while establishing a more general property of the MPME in this environment.

**Proposition 1** There does not exist a MPME with \( \mathcal{X}_B = 0 \) for all \( B > \hat{B} \).

Thus, we rule out the existence of equilibria where the government perfectly smoothes distortions across periods for all levels of debt—see (24). Consequently, in any MPME the allocation of the day-good is necessarily distorted for some levels of debt.

Given the above results, we can proceed with the analysis of the equilibrium. From (23) we have \( \lambda = \frac{\varepsilon_x}{\alpha} - 1 \). After some substitutions, the system of equations characterizing the MPME is:

\[
\begin{align*}
\beta f'_y v'(v_g - v'_g) + (v_g - \alpha)\varepsilon_x \mathcal{X}_B' &= 0 \quad (25) \\
\alpha \eta (u_x - f_y) - (v_g - \alpha)(f_y + f_{yy}) (1 + B) + \alpha \zeta (u_{xx} - \frac{\eta f_{yy}}{1-\eta}) &= 0 \quad (26) \\
v_g (U_c - \alpha) + (v_g - \alpha)U_{cc} c &= 0 \quad (27) \\
(U_c - \alpha) c - \alpha g + \beta \eta x'(u'_x - f'_y) + \beta f'_{yy} x'(1 + B') - f_y x (1 + B) &= 0, \quad (28)
\end{align*}
\]

\( u_x - f_y \geq 0, \quad \zeta \geq 0 \) and \( \zeta(u_x - f_y) = 0 \).

Some results are intractable for general specifications, as they depend on the properties of the third derivatives of some functions. The following assumptions are sufficient to provide the desired results.

**Assumption 2** The following conditions hold: (i) \( 2f_{xx} + f_{xxx} \geq 0 \) for \( x \in (0, \hat{x}] \); and (ii) \( U_{cc} c - (U_c - \alpha)(1 + \frac{U_{ccc}}{U_{cc}}) < 0 \) for \( c \in (0, \hat{c}] \).

Note that these requirements are quite weak. For example, both are satisfied if we assume that \( f(x) = \frac{-1}{\gamma} \) and \( U(c) = \frac{c^{\gamma-\rho}}{1-\rho} \), \( \gamma, \rho > 0 \), a common assumption.

**Proposition 2** A Markov-Perfect Monetary Equilibrium (MPME) is characterized by critical debt levels \( \hat{B} < \hat{B}^\infty < \ldots < \hat{B}^0 < -1 < B^* \), such that:

(i) \( \mathcal{X}(B) = \hat{x} \) for all \( B \leq -1 \) and \( \mathcal{X}(B) < \hat{x} \) for all \( B > -1 \).

(ii) \( \mathcal{C}(B) = \hat{c} \) and \( \mathcal{G}(B) = \hat{g} \) for all \( B \leq \hat{B} \); and \( \mathcal{C}(B) < \hat{c} \) and \( \mathcal{G}(B) < \hat{g} \) for all \( B > \hat{B} \).

(iii) \( \mathcal{B}(B) = \frac{\alpha g}{\beta u_x x} + \frac{1+B}{\beta} - 1 \) for all \( B \leq \hat{B} \).

(iv) \( \mathcal{B}(B) = B \) for all \( B \in [\hat{B}, \hat{B}^\infty] \).
Proposition 2— and is characterized by $B^*$ and is continuous across periods. The proof of Proposition 2 is straightforward: (i) tax rates are zero for all $B \leq \tilde{B}$ and positive for all $B > \tilde{B}$; and (ii) monetary policy is non-distortionary for all $B \leq -1$ and distortionary for all $B > -1$. Figure 1 provides a more complete characterization of the MPME, using numerical methods to solve the equilibrium.\footnote{The numerical computation of Markov-perfect equilibria for dynamic policy games has been described extensively elsewhere. The method used here follows the projection algorithm described in Martin (2009) with the caveat that one needs to account for the (countable) discontinuities in policy. The code is available upon request.}

The first-best is implemented for any $B \leq \tilde{B}$. The “kink” in $X$ at $B = -1$ caused by the non-negativity constraint (18), introduces some discontinuities in government policy for $B \in (\tilde{B}, \tilde{B}_0)$. Specifically, debt is kept constant for some initial range and is a discontinuous function afterwards. There is an infinite and uncountable number of steady states and an infinite but countable number of critical points at which debt “jumps”. As analyzed above, if $X' = 0$, the current government will equate today’s and tomorrow’s distortions. For $B \in (\tilde{B}, \tilde{B}_0]$, the government perfectly smooths distortions across time, even though it does not implement the first-best. For $B \in (\tilde{B}_0, \tilde{B}^\infty]$, the government equates distortions across a finite number of periods, at the cost of increasing debt. Eventually, debt becomes large enough (i.e., government claims on the private sector become small enough) that the current government expects future governments to start distorting the allocation of the day-good—the incentives to inflate grow with debt—and thus, no longer has incentives to perfectly smooth distortions across periods.

For $B \geq -1$, the MPME is well-behaved (in the sense that there are no discontinuities in policy stemming from fundamentals) and features a unique steady state, $B^*$. Furthermore, the money growth rate is always above the Friedman rule. Note also that $B(B) > -1$ for all $B \geq -1$. Thus, for positive applications of the theory we can focus on $B \geq -1$.

### 4.3 Long-run debt

In this section, we analyze the properties of $B^*$. This steady state features $x^* = 0$—see (24) and the proof of Proposition 2—and is characterized by $\{x^*, c^*, g^*\}$ satisfying

$$v_x^*(u_x^* - f_y^*) + (v_y^* - \alpha)(u_{xx}^* x^* - f_{yy}^* y^*) = 0 \quad (29)$$

$$v_x^*(U_{c}^* - \alpha) + (v_y^* - \alpha)U_{cc}^* c^* = 0 \quad (30)$$

$$(U_c^* - \alpha)c^* - \alpha g^* + \eta x^*(u_x^* - f_y^*) \left( \beta - \frac{\alpha (1 - \beta) f_y^*}{(v_y^* - \alpha)(f_y^* + \alpha g^*)} \right) = 0, \quad (31)$$

$$B(\tilde{B})^0 = -1. \text{ There exist critical points } \tilde{B}^j = (1 + \tilde{B}^0)^{\frac{\beta - 1}{1 - \beta}} - 1, \ j = 0, \ldots, \infty, \text{ such that:}$$

- $B$, $C$ and $G$ are discontinuous at these critical points; $B(\tilde{B})^j = \tilde{B}^{j-1}$, $C(\tilde{B})^j = C(\tilde{B}^0)$ and $G(\tilde{B})^j = G(\tilde{B}^0)$ for all $j = 1, \ldots, \infty$.

- $B(B) > -1$ and $X'_B < 0$ for all $B \geq -1$.

- $B(B^*) = B^*$. 

$B(\tilde{B}_0) = 12$. This steady state features $x^* = 0$—see (24) and the proof of Proposition 2—and is characterized by $\{x^*, c^*, g^*\}$ satisfying
Figure 1: Markov-Perfect Monetary Equilibrium
where \( y^* = \frac{y^*}{1-\eta} \), and
\[
B^* = -\frac{\eta(u_x^* + u_{xx}^*x^*)}{f_y^* + f_{yy}y^*} - 1 + \eta.
\]
(32)

Figure 1 suggests this steady state is stable, which makes the theory suitable for positive analysis. For a limiting case, we can prove this property.

**Proposition 3** Assume \( v_g = \psi g^{-\nu}, \psi > \alpha, \nu > 0 \). Then, as \( \nu \to 0 \), \( B(B) \to B^* \) for all \( B \geq -1 \).

Countries typically feature positive amounts of government debt. The following proposition relates long-run debt to the model’s fundamentals.

**Proposition 4** Properties of \( B^* \): (i) \( B^* > 0 \) only if \(-\frac{u^*_x}{u^*_x} > 1\); (ii) as \( \eta \to 1 \), \( B^* > 0 \) if and only if \(-\frac{u^*_x}{u^*_x} > 1\); (iii) as \( \eta \to 0 \), \( B^* \to -1 \); and (iv) if \( u_x = x^{-\sigma}, \sigma > 0, \) and \( f_x = \phi > 0 \), then \( \eta \sigma > 1 \) is a sufficient condition for \( B^* > 0 \).

As we can see, critical for long-run debt are the curvature of the utility function for the day-good, \( u(x) \), and the measure of buyers in the day market, \( \eta \). Given part (i) of Proposition 4, if \( u(x) \) features a constant intertemporal elasticity-of-substitution, then its curvature needs to be higher than log for debt to be positive in the long-run (this condition is also sufficient as the measure of buyers approaches one—part (ii)). This result is caused by the fact that the curvature in \( u \) determines how distortionary the inflation tax is. A high curvature in \( u \) implies inflation is very costly to the agent, which means the (benevolent) government has low incentives to use inflation and thus, high incentives to push the burden of taxation to the future, i.e., increase debt. The measure of buyers plays a crucial role as well, since it weights the distortion costs caused by inflation. Part (iii) shows that as this weight converges to zero, inflation is heavily used to reduce the financial burden of debt, which converges to \(-1\) in the long-run. Part (iv) provides a sufficient condition for positive long-run debt, for typically used functional forms.

One implication of Proposition 4 is that under the parameterization of Lagos and Wright (2005) and most of the subsequent literature, which feature \( \sigma < 1 \), the economy would converge to negative government debt in the long-run.

### 4.4 Liquid or illiquid bonds?

Positive long-run debt is not only relevant for the analysis of actual economies, but also necessary for illiquid bond to be socially optimal. To see this, note that since money is neutral in this environment, issuing only liquid bonds—i.e., bonds that are perfect substitutes of money—is equivalent to setting end-of-period debt to zero. Suppose we allow the government to choose whether to issue either liquid or illiquid bonds, but assume it cannot change the liquidity properties of inherited debt.\(^\text{13}\) This option is equivalent to choosing between zero debt and some other level, a decision which is already accounted for in the government’s problem. Assuming \( B(B) \) is increasing in \( B \) and

\(^{13}\)If the government could make illiquid bonds liquid at the beginning of the period, it would always choose to do so since it amounts to a non-distortionary default on debt.
parameters are such that \( B^* > 0 \), then \( B(B) > 0 \) for all \( B \geq 0 \). In other words, the government prefers to issue illiquid bonds.

A related question is what institution society would select. Suppose we start with zero debt and allow society to commit to either having liquid or illiquid bonds. The choice now is between the MPME as characterized above and the allocation associated with zero debt in every period. If parameters are such that \( B^* > 0 \), then \( B(0) > 0 \), i.e., at zero debt the government always prefers to issue some. Since the government is benevolent, this means that agents’ welfare is improved if we allow the government to issue illiquid bonds. Thus, society would choose to make government bonds illiquid or, equivalently, impose a cash-in-advance constraint in the day market.

The results here are related to Kocherlakota (2003), where illiquid bonds are essential since they allow agents to trade money intertemporally. In contrast, here, illiquid bonds are optimal since they allow the government to trade-off distortions across periods.

### 4.5 Time-consistency and commitment

As mentioned above, we have \( \varepsilon_{x'} = 0 \). That is, even though small changes in debt choice at \( B^* \) have an effect on future policy (given that \( X_B' < 0 \)), the positive and negative effects of these changes on the current government budget constraint are balanced out. In other words, the time-consistency problem, which is driving the change in debt, cancels out at the steady state. It follows that if the governments starts at \( B^* \), it will stay there, regardless of its ability to commit. The following statement formalizes this argument.

**Proposition 5** Suppose initial debt is equal to \( B^* \). Then a government with commitment and a government without commitment will both implement the allocation \( \{x^*, c^*, g^*\} \) and choose debt level \( B^* \) in every period.

This result has important implications for institutional reform as it shows that endowing a government at \( B^* \) with a commitment technology would have no effect on policy.\(^{14}\) It also shows that time-consistency of policy under commitment is not necessarily linked to optimality of the Friedman rule (e.g., see Alvarez, Kehoe and Neumeyer, 2004). The key elements for the result in Proposition 5 are two: first, the government weights current and future distortions equally (\( \varepsilon_B' = -\beta \varepsilon_B' \)), which is a standard feature; and second, the time-consistency problem is internalized through a single good—the term \( \varepsilon_{x'} X_B' \) in (24). One way to break this result, without abandoning the Lagos-Wright framework, is to assume non-separable preferences in \( x \) and \( c \). In this case, the extra term \( \varepsilon_{c'} C_B' \) would appear in the GEE and typically, we would not obtain \( \varepsilon_{x'} = \varepsilon_{c'} = 0 \).

Proposition 5 begs the question of how the MPME and the Ramsey policy compare for any level of initial debt. The derivation of the Ramsey policy is well understood and thus, omitted here.\(^{15}\) Figure 2 compares debt and monetary policy with and without commitment, as a function of initial debt, \( B_0 \in [\hat{B}, 2B^*] \), using the same parameterization as in Figure 1. Graph with tax rates and expenditure are omitted to save space. With commitment, both taxes and government

---

\(^{14}\)Note however, that a commitment technology would still significantly affect how policy reacts to shocks.

\(^{15}\)See the proof of Proposition 5 for the formulation of the Ramsey problem and some basic results. See also Aruoba and Chugh (2008) for a formulation and analysis of the Ramsey problem in a similar environment.
expenditure are constant in every period and a function of initial debt. In section 5.3, I evaluate the welfare differences of these two regimes, for a calibrated economy.

The left panel of Figure 2 compares debt policy. As is typically the case in stationary environments, a government with commitment changes debt in the first period and never again. Thus, there are two debt policies, one for the initial period and one for all other periods (the 45-degree line). Note that any level of debt can be supported in the long-run, independent of the environment fundamentals. There are three debt levels where long-run debt is equal to initial debt (regardless of parameterization): $\hat{B}_0$, $-1$ and $B^*$. Also worth mentioning is that the Ramsey government reduces debt in the first period for $B_0 \in (\hat{B}_0, -1)$, which contrasts with the policy in the MPME. The difference in behavior is due to the Ramsey government not implementing the Friedman rule for any $B_0 > \hat{B}_0$, as explained below.

The right panel of Figure 2 compares monetary policy. The Ramsey planner features two policies: one for the initial period and one for all other periods, both shown as function of initial debt. One important feature is that $\hat{x}$ is never implemented after the initial period for $B_0 > \hat{B}$, a fact also highlighted in Aruoba and Chugh (2008). Thus, even though $\hat{x}$ is implemented in the initial period for all $B \leq -1$, the Friedman rule is never implemented for any $B_0 > \hat{B}$. In contrast, a government that lacks commitment implements the Friedman rule for all $B \leq \hat{B}^0$. An important difference then is that a Ramsey government typically distorts the economy using both fiscal and monetary instruments, whereas a Markov government uses both distortions only when net nominal obligations are positive.

5 Parameterized Analysis

I will now rely on specific functional forms to conduct some analytical and numerical analysis. First, I will make sufficient assumptions to solve the model analytically and perform some steady
state comparative statics. Second, I will show a calibration that matches selected statistics for the U.S. economy. Third, I use this calibration to evaluate the welfare loss due to lack of commitment. Throughout this section, the following functional forms hold.

**Assumption 3**

\[ u(x) = \frac{x^{1-\sigma-1}}{1-\sigma}, \quad f(x) = \phi x, \quad U(c) = \psi^{1-\rho-1-1\rho-1}, \quad \sigma, \phi, \rho, \psi, \nu > 0. \]

### 5.1 Analytical characterization

Using suitable assumptions on parameters, we can characterize the MPME analytically and perform some comparative statics at \( B^* \). Although not a general proof, this example establishes existence of the equilibrium for a limiting case.

To allow for an analytical solution, let \( \rho \to 1 \) and \( \nu \to 0 \). This last assumption enables us to use the result from Proposition 3. Note that as \( \nu \to 0, \hat{g} \to \infty \); thus, we focus on \( B \geq -1 \). Since now \( v \to \psi \), we need to make additional assumptions to ensure the solution is interior. Let \( 1 - \frac{1}{\sigma} < \frac{\alpha}{\psi} < \beta \). With these assumptions, the MPME converges to:

\[
\begin{align*}
B(B) &= \frac{\eta \alpha \sigma}{\psi - \sigma(\psi - \alpha)} - 1 \\
\chi(B) &= \left( \frac{\eta \alpha}{\phi(\eta \alpha + (\psi - \alpha)(1 + B))} \right)^{\frac{1}{\sigma}} \\
C(B) &= \frac{1}{\psi} \\
G(B) &= \frac{1}{\alpha} \left\{ 1 - \frac{\alpha}{\psi} + \beta \eta \sigma \left( \frac{\phi \psi}{\psi - \sigma(\psi - \alpha)} \right)^{\frac{\sigma - 1}{\sigma}} - (1 + B) \left( \frac{\eta \alpha}{\phi(\eta \alpha + (\psi - \alpha)(1 + B))} \right)^{\frac{1}{\sigma}} \right\}
\end{align*}
\]

for all \( B \geq -1 \).

**Proposition 6** Assuming \( 1 - \frac{1}{\sigma} < \frac{\alpha}{\psi} < \beta \), as \( \rho \to 1 \) and \( \nu \to 0 \) we get the following comparative statics results at \( B^* \).

(i) An increase in the disutility from labor \( \alpha \): (1) decreases taxes and inflation; (2) decreases debt if \( \sigma > 1 \), increases debt if \( \sigma < 1 \) and has no effect on debt if \( \sigma = 1 \).

(ii) An increase in the preference for the public good \( \psi \): (1) increases taxes and inflation; (2) increases debt if \( \sigma > 1 \), decreases debt if \( \sigma < 1 \) and has no effect on debt if \( \sigma = 1 \); (3) increases government expenditure.

(iii) A decrease in the intertemporal elasticity of substitution for the day-good, i.e., an increase in \( \sigma \): (1) has no effect on taxes; (2) increases debt and inflation; (3) increases government expenditure if \( \phi > \kappa \), decreases expenditure if \( \phi < \kappa \) and has no effect on expenditure if \( \phi = \kappa \), where \( \kappa \equiv \frac{\psi - \sigma(\psi - \alpha)}{\psi \psi - \sigma(\psi - \alpha)} \).

(iv) An increase in the measure of day-good consumers \( \eta \): (1) has no effect on taxes; (2) increases inflation, debt and government expenditure.
(v) An increase in the marginal cost of the day-good, $\phi$: (1) does not affect taxes, inflation or debt; (2) increases government expenditure if $\sigma > 1$, decreases expenditure if $\sigma < 1$ and has no effect on expenditure if $\sigma = 1$.

Note several features from the proposition above. First and foremost, if $\sigma > 1$ (a necessary condition for $B^* > 0$), debt and inflation react in the same qualitative way to changes in parameters. Second, increasing the disutility from labor has the same qualitative effect as decreasing the preference for the public good. The implication is that, for numerical work, we can safely normalize $\psi$ to 1. Third, decreasing the intertemporal elasticity of substitution has the same qualitative effect (with some caveats for expenditure) as increasing the measure of day-good consumers. Coupled with part (iv) of Proposition 4, this result suggest that a steady state can be calibrated with different combinations of values for $\sigma$ and $\eta$. Fourth, long-run debt and inflation are increasing in $\sigma$ and $\eta$. Fifth, changes in the marginal cost of producing the day-good have no effect on long-run government policy, except for the level of expenditure.

5.2 Calibration

The steady state $B^*$ can be easily calibrated to an actual economy. The model has definite predictions for the following policy variables: debt, inflation, taxes, interest rate, expenditure and velocity of circulation. All these variables enter the government budget constraint (indirectly in the case of velocity); thus, we cannot target all of them since the constraint needs to be satisfied. I will take the tax rate $\tau$ as the residual. This leaves 5 targets.

Under Assumption 3, the model is overidentified. Thus, set $\psi = \nu = 1$, i.e., $\nu(g) = \ln g$, and $\eta = 0.5$. The parameters left to calibrate are then: $\alpha$, $\beta$, $\phi$, $\rho$ and $\sigma$. Table 1 summarizes the parameter choice and calibration targets.

<table>
<thead>
<tr>
<th>Target statistics</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B(1 + \mu)/Y$</td>
</tr>
<tr>
<td>0.310</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameter values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$, $\beta$, $\phi$, $\rho$, $\sigma$</td>
</tr>
<tr>
<td>4.920, 0.982, 1.119, 9.040, 5.212</td>
</tr>
</tbody>
</table>

Note: $\nu = \psi = 1$, $\eta = 0.5$.

All calibration targets are taken from U.S. annual data for the period 1962-2006. First, define nominal GDP as the sum of nominal output in the day and night markets. Abusing notation slightly, let $Y$ be nominal GDP normalized by the aggregate money stock, i.e., $Y = \eta \tilde{p}x + p(c + g)$. Let $C \equiv pc$ and $G \equiv pg$ and recall that in equilibrium, $\tilde{p}x = 1$. Given $\eta = 0.5$, we get $Y = 0.5 + C + G$. Note that by the equation of exchange, velocity of circulation is defined as the nominal GDP divided
by the aggregate money stock. Thus, the first target is to set $Y$ equal to the velocity of circulation in the data. Following the literature, take $M_1$ as the measure of money, which implies velocity is 6.3.

In steady state, the inflation rate is equal to the money growth rate, $\mu$. Using the CPI as the measure of the price level, the inflation rate for the period averaged 4.4% annual.

The third target is debt over GDP. In the data, government debt is measured at the end of the period. Thus, the relevant numerator is $B(1 + \mu)$. Since $B$ is the bond-to-money ratio, debt over GDP is given by $\frac{B(1 + \mu)}{Y}$. In the U.S., debt over GDP, excluding holdings by federal agencies and the Federal Reserve Banks, averaged 31% between 1962 and 2006. Given that we are targeting $\mu = 0.044$ and $Y = 6.3$, this implies a target for $B$ roughly equal to 1.871.

Evaluating (14) and (17) in steady state, we get $q(1 + \mu) = \beta$. Note that $q$ is the inverse of the gross nominal interest rate. Take the 1-year treasury constant maturity rate, which averaged 6.3% annual in the period considered. Thus, $\beta = 1.044 \times 1.063 \approx 0.982$.

The last target is government expenditure. In the model, $G$ represents nominal government expenditure normalized by the aggregate money stock. In the data, federal government outlays, net of debt interest payments (which the model accounts for in the discounted price of bonds, $q$), averaged 18% per year in terms of GDP. Thus, we set $C = 0.18$ or, equivalently, $G = 1.134$. This also allows us to reexpress the target for velocity. We have $Y = 0.5 + C + G = 6.3$. Given, $Y = 6.3$ and $G = 1.134$, this implies $C = 4.666$.

The strategy now is to choose $\alpha$, $\phi$, $\rho$ and $\sigma$ so that solving (29)–(32) implies $B = 1.871$, $\mu = 0.044$, $C = 4.666$ and $G = 1.134$. Note that using (1), we can write labor taxes in terms of the calibration targets: $\tau = \frac{(1 - \beta)B - \mu + G}{C + G} \approx 0.194$.

### 5.3 Welfare

We can use the calibration to evaluate the welfare loss due to lack of commitment in this environment. Specifically, given initial debt $B_0$, I calculate the one-time fee that the agent is willing to pay to switch from the MPME to the Ramsey policy, expressed in terms of period-consumption. That is, the function $\Delta(B_0)$ that satisfies

$$
\eta u(\mathcal{X}(B_0)(1 + \Delta(B_0))) - (1 - \eta) f \left( \frac{\eta \mathcal{X}(B_0)}{1 - \eta} \right) + U(\mathcal{C}(B_0)(1 + \Delta(B_0)))
- \alpha(\mathcal{C}(B_0) + \mathcal{G}(B_0)) + v(\mathcal{G}(B_0)) + \beta \mathcal{V}(\mathcal{B}(B_0)) = V^R(B_0),
$$

where $V^R(B_0)$ is the present value utility under the Ramsey policy, given initial debt $B_0$.

Table 2 shows the equivalent compensation measure $\Delta$ for selected debt levels. As a reference, the table includes the corresponding level of debt over GDP, measured at the beginning of the period. Given the calibration parameters, $\hat{B}^\infty$ actually features the highest $\Delta$: 0.43%. Note that this is not a general result (e.g., using the parameterization of section 4.2, $\Delta$ peaks somewhere in between $\hat{B}^\infty$ and $\hat{B}^0$). For more empirically relevant debt levels, say between 0 and 2$B^*$, $\Delta$ is at most 0.04%, which is quite low when compared to typical welfare measures in macroeconomics (e.g., the cost of business cycles or the cost of 10% annual inflation).
Table 2: Welfare loss due to lack of commitment

<table>
<thead>
<tr>
<th>$B$</th>
<th>$B/Y$</th>
<th>$\Delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B$</td>
<td>-9.99</td>
<td>0.00%</td>
</tr>
<tr>
<td>$\tilde{B}^\infty$</td>
<td>-2.31</td>
<td>0.43%</td>
</tr>
<tr>
<td>$\tilde{B}^0$</td>
<td>-0.21</td>
<td>0.12%</td>
</tr>
<tr>
<td>$-1$</td>
<td>-0.17</td>
<td>0.10%</td>
</tr>
<tr>
<td>0</td>
<td>0.00</td>
<td>0.04%</td>
</tr>
<tr>
<td>$B^*$</td>
<td>0.30</td>
<td>0.00%</td>
</tr>
<tr>
<td>$2B^*$</td>
<td>0.57</td>
<td>0.04%</td>
</tr>
</tbody>
</table>

6 Concluding remarks

The model presented in this paper features some attractive properties for further policy analysis. The steady state with positive net nominal government liabilities is unique, highly tractable and features positive taxes and inflation above the Friedman rule. It is straightforward to verify that these properties would survive a number of extensions to the basic environment; e.g., trading frictions in the day market and productivity and/or government expenditure shocks. Whether these extensions have any implications for the determination of government policy is work currently in progress.
A Proofs

A.1 Proof of Lemma 1

Suppose $\lambda = 0$. Then from (22) and (23), we get $c = \hat{c}$ and $g = \hat{g}$; from (21) we get $u_x - f_y = \zeta = 0$, i.e., $x = \hat{x}$. From (20) $\lambda' = 0$, which from (21)—(23) implies $x' = \hat{x}, \ c' = \hat{c}$ and $g' = \hat{g}$. Suppose now that $x = x', c = \hat{c}$ and $g = g' = \hat{g}$. Then, equations (21)—(23) all imply $\lambda = 0$.

A.2 Proof of Proposition 1

Suppose $X_B = 0$, i.e., $x$ is constant for all $B$ in a MPME. From (24), $X_B = 0$ implies $\lambda = \lambda'$. At $B = 1, \varepsilon_x = 0$ and so (21) becomes $\eta(u_x - f_y) + \zeta(u_xx - \frac{\eta f_{yy}}{1-\eta}) = 0$. The only solution is $x = \hat{x}, \zeta = 0$. Thus, $X(B) = \hat{x}$. Consider now $B > -1$. Since $x = \hat{x}$, (21) is $-\lambda(1+B)(f_y + \hat{f}_y \hat{g}) + \zeta(\hat{u}_xx - \frac{\eta f_{yy}}{1-\eta}) = 0$, where $\lambda = \zeta = 0$ is the only solution. Thus, $c = \hat{c}, g = \hat{g}$ and from (19), $B(B) = \frac{a_\alpha}{\beta \alpha x} + \frac{1+B}{\beta} - 1$ for all $B > -1$. If we plug this debt function into (19) for $B \leq -1$, we get that $c = \hat{c}, g = \hat{g}$, $\lambda = \zeta = 0$ is also a solution to (20)—(23). Thus, the MPME features the first-best for all $B$. The implied policy is $\hat{\mu} = \beta - 1, \hat{\tau} = 0, \hat{q} = 1$ and $\hat{p} = \frac{\alpha}{\beta u_x}$ for all $B$. The value function is $V(B) = \hat{V} \equiv \frac{nu(\hat{x}) + (1-\eta)f(\hat{g}) + U(\hat{c}) - \alpha(\hat{c} + \hat{g}) + \hat{v}(\hat{g})}{1-\beta}$.

A day-market consumer arrives to the night market with nominal balances equal to $B$ and works $\hat{n}^c = \frac{\hat{c}}{1-\tau} = \frac{B - (1+\hat{\mu})(m' + \hat{\mu})}{\hat{\mu}(1-\tau)} = \frac{\hat{c} - \hat{u}_x \hat{\hat{x}} (B - \beta (m' + \hat{\mu}))}{\alpha}$. If he does not deviate, he chooses $m' = 1$ and $b' = B(B)$ and hence works $\hat{n}^c = \hat{c} + \hat{g} + \hat{u}_x \hat{x}$. Thus, the equilibrium pay-off for a consumer at night is $\hat{W}^c = U(\hat{c}) - \alpha(\hat{c} + \hat{g}) - \hat{u}_x \hat{x} + \hat{v}(\hat{g}) + \beta \hat{V}$.

Consider now a consumer that deviates at night in the following way: he still consumes $\hat{c}$ and chooses $m' = 1$, but now he sells all his bonds, i.e., $b' = 0$ and saves on work accordingly. After the current period, the agent maintains a portfolio of zero bonds and finances his (first-best) consumption with fiat money and labor only. Thus, in future periods the agent works $\hat{n}_d = \hat{c} + \hat{u}_x \hat{x} (1-\beta) - \hat{f}$ Expected nominal balances $z$ are equal to 1, and thus expected labor is $\hat{c} - \frac{\hat{u}_x \hat{x} (1-\beta)}{\alpha}$. The value of this continuation strategy is then $V_d = \frac{nu(\hat{x}) + (1-\eta)f(\hat{g}) + U(\hat{c}) - \alpha(\hat{c} + \hat{g}) + \hat{v}(\hat{g})}{1-\beta}$. The deviation period, the consumer chooses $c = \hat{c}, m' = 1$ and $b' = 0$ and thus, works $\hat{n}^c = \hat{c} + \hat{u}_x \hat{x} (1-\beta) - \hat{f}$. The pay-off from deviating is $W_d^c = U(\hat{c}) - \alpha(\hat{c} - \hat{u}_x \hat{x} (1-\beta) + \hat{f} + \beta V_d).$ A consumer has an incentive to deviate only if $W_d^c > \hat{W}^c$. After some simple algebra, this condition reduces to $B > \hat{B}$. Thus, there is a profitable deviation from the equilibrium if debt is sufficiently high. Given $B(B) > B$ for all $B > \hat{B}$, agents have an incentive to deviate if the bond-money ratio is above $\hat{B}$. Thus, $X_B = 0$ for all $B > \hat{B}$ cannot be a MPME.

A.3 Proof of Proposition 2

A.3.1 Parts (ii) and (iii)

First note that regardless of $B$, if $\lambda = 0$ then by Lemma 1 the first-best is being implemented. In this case, the government budget constraint (19) implies $B' = \frac{a_\alpha}{\beta \alpha u_x} + \frac{1+B}{\beta} - 1$. For $B > \hat{B}$,
we have \( B' > B \), which implies that, in all periods, debt is rolled-over while the government implements the first-best. From the proof of Proposition 1, this cannot be an equilibrium. However, \( B' = \frac{\alpha_g}{\beta u_g x} + 1 + \frac{B}{\beta} - 1 \) for \( B \leq \hat{B} \) is admissible and, since it maximizes agents welfare, is the equilibrium policy. Thus, \( \lambda = 0 \) only for \( B \leq \hat{B} \) and we get (iii). Given \( \lambda > 0 \) for all \( B > \hat{B} \), part (ii) follows from equations (22) and (23).

A.3.2 Part (i)

Consider (21), i.e., the first-order condition with respect to \( x \). From part (ii), \( x = \hat{x} \) for \( B \leq \hat{B} \). Now focus on \( B > \hat{B} \), i.e, \( \lambda > 0 \). If \( B < -1 \) then \( \eta(u_x - f_y) - \lambda(1 + B)(f_y + f_y y) > 0 \) and thus, \( \zeta > 0 \), which implies \( x = \hat{x} \). If \( B = -1 \) then \( u_x - f_y = \zeta = 0 \) and thus, \( x = \hat{x} \). If \( B > -1 \) then \( u_x - f_y > 0 \), \( \zeta = 0 \) and thus, \( x < \hat{x} \). The assumptions on \( u \) and \( f \) guarantee that an interior solution for \( x \) to \( \eta(u_x - f_y) = \lambda(1 + B)(f_y + f_y y) \) exists.

A.3.3 Auxiliary lemmas

**Lemma 2** \( B(B) \) is an injection.

**Proof.** Suppose not, i.e., there exist debt values \( B^0 < B^1 \) such that \( B(B^0) = B(B^1) \). Note that this implies \( \mathcal{X}(B(B^0)) = \mathcal{X}(B(B^1)) \) and \( \mathcal{G}(B(B^0)) = \mathcal{G}(B(B^1)) \). Thus, (25) implies \( \mathcal{G}(B^0) = \mathcal{G}(B^1) \) and so, (27) implies \( \mathcal{C}(B^0) = \mathcal{C}(B^1) \). The government budget constraint can thus be written as \( f_y x(1 + B) = K \), where \( K \) does not depend on \( x \) or \( B \). There are three cases to consider: \( B^0 < B^1 \leq -1 \); \( B^0 \leq -1 < B^1 \); and \( -1 < B^0 < B^1 \). If \( B^0 < B^1 \leq -1 \) then from part (i), \( \mathcal{X}(B^0) = \mathcal{X}(B^1) = \hat{x} \) and thus, the government budget constraint cannot be satisfied simultaneously for \( B^0 \) and \( B^1 \). If \( B^0 \leq -1 < B^1 \) then \( f_y x(1 + B^0) \leq 0 \) and \( f_y x(1 + B^0) > 0 \); thus, the government budget constraint cannot be satisfied at both debt levels. If \( -1 < B^0 < B^1 \), then replace \( f_y x(1 + B) = K \) in (26) and get \( \alpha \eta(u_x - f_y) = \frac{(v_g - \alpha)(f_y + f_y y) K}{f_y x} = 0 \), i.e., \( \mathcal{X}(B^0) = \mathcal{X}(B^1) \); then, the government budget constraint cannot be satisfied for both \( B^0 \) and \( B^1 \), a contradiction.

**Lemma 3** \( c \) changes in the same direction as \( g \).

**Proof.** We focus on \( B > \hat{B} \), since for \( B \leq \hat{B} \), both \( c \) and \( g \) are constant. Thus, \( \lambda > 0 \) from part (ii). Take (27) and write it as \( F(c, g) = 0 \). By the Implicit Function Theorem, \( \frac{dc}{dg} = -\frac{F_g}{F_c} \). We get \( F_g = v_g g(U_c - \alpha + U_{cc} c) > 0 \), since from (22), \( \lambda > 0 \) implies \( \varepsilon_c < 0 \), i.e., \( U_c - \alpha + U_{cc} c < 0 \). Thus, to show \( \frac{dc}{dg} > 0 \) we need \( F_c < 0 \). We get \( F_c = v_g U_{cc} + (v_g - \alpha)(2U_{cc} + U_{ccc} c) \), which using (27) can be rewritten as \( F_c = U_{cc} - (U_c - \alpha)(\frac{1}{\varepsilon_c} + \frac{U_{ccc}}{U_{cc}}) < 0 \), given Assumption 2.

**Lemma 4** For \( B > \hat{B} \), \( \beta f_y' x' + \varepsilon_x' \lambda' B > 0 \), i.e., increasing \( B' \) relaxes the government budget constraint.

**Proof.** By Lemma 1 and part (ii), for \( B > \hat{B} \) we have \( \lambda, \lambda' > 0 \). Rearrange (24) as \( \lambda(\beta f_y' x' + \varepsilon_x' \lambda' B) - \lambda' f_y' x = 0 \); thus, \( \beta f_y' x' + \varepsilon_x' \lambda' B > 0 \).
We now show part (v). Define $\tilde{B}^0 \in (\hat{B}, -1)$ such that $B(\tilde{B}^0) = -1$; by Lemma 2 there is at most one such debt level. At $\tilde{B}^0$, we have $x = x' = \hat{x}$ and $\varepsilon x' = \beta \eta (\hat{u}_{xx} \hat{x} - \hat{f}_{yy} \hat{y}) < 0$. Differentiating (26) with respect to $B$ and evaluating this expression at $B = -1$ we get: $\alpha \eta (\hat{u}_{xx} - \frac{\hat{f}_{xx}}{\hat{u}_{xx}}) \hat{X}_B - (v_g - \alpha)(\hat{f}_y + \hat{f}_{yy} \hat{y}) = 0$. Thus, $X_B < 0$ at $B = -1$, i.e., $X'_B < 0$ at $B = \tilde{B}^0$. Then, from (28)
\[
\tilde{B}^0 = \frac{(\hat{U}_c - \alpha) \hat{c} - \alpha \hat{g}}{\hat{u}_{x\hat{x}}} - 1,
\]
where $\{\hat{c}, \hat{g}\}$ solve (25) and (27). Using the expression for $\hat{X}_B$ derived above, we can simplify this system of equations. We get
\[
\frac{\hat{v}_g - \alpha}{v'_g - \alpha} - 1 + \frac{\hat{v}_g - \alpha}{\alpha} (1 + \frac{\hat{f}_{yy} \hat{y}}{\hat{f}_y}) = 0
\]
\[
\hat{v}_g (\hat{U}_c - \alpha) + (\hat{v}_g - \alpha) \hat{u}_{cc} \hat{c} = 0,
\]
where “primes” denote the equilibrium values at $B = -1$. We now verify that $\tilde{B}^0 \in (\hat{B}, -1)$. First, $\tilde{B}^0 > \hat{B}$ follows from part (iii), since $B' = -1$ is not a solution to $B(B)$ for $B \leq \hat{B}$. Second, since $v'_g - \alpha > 0$ from part (ii), the first equation above implies $\hat{g} > \hat{g} > g'$, i.e., $0 < \hat{\lambda} < \lambda'$. Lemma 3 implies $c > \hat{c} > c'$. Since compared to $B = -1$, the government at $\tilde{B}^0$ is implementing a higher utility allocation, $\hat{x} = x'$, $\hat{c} > c'$, $\hat{g} > g'$, and is less constrained ($0 < \hat{\lambda} < \lambda'$ and $\hat{c} = c' = 0$), it must be that the financial burden is lower, i.e., $\tilde{B}^0 < -1$.

Notice that if we approach $B'$ from the left of $-1$, $X'_B = 0$ from part (i). Thus, consider approaching $B' = -1$ from the left at $B = \tilde{B}^0$. Then, $X'_B = 0$ and (25) implies $g = g' < \hat{g}$ and hence, $c = c' < \hat{c}$ from (27). But then, the government budget constraint (28) cannot be satisfied at $\tilde{B}^0$ since it is satisfied for $\{\hat{c}, \hat{g}\}$, as described above. Given that the solution approaching from the left and right are different, the equilibrium is discontinuous in $B, C, G$ (not $X$, since by part (i) it is flat in this range).

Having identified $\tilde{B}^0$ and the discontinuity of the MPME at this debt level, focus on $B \in (\hat{B}, \tilde{B}^0)$ and such that $B(B) < -1$ (we will verify this property below). Consider (24), i.e., $\beta f'_y x' (\lambda - \lambda') + \lambda x'_B X'_B = 0$. Since $B \in (\hat{B}, \tilde{B}^0)$ and $B(B) < -1$, we have $\lambda, \lambda' > 0$ from part (ii), and $x = \hat{x}$ and $X'_B = 0$ from part (i). Therefore, (24) implies $\lambda = \lambda' > 0$. Thus, either $B' = B$ and we are in (a distortionary) steady state, or $\{c = c' < \hat{c}; g = g' < \hat{g}\}$, while debt changes. Suppose we are not in steady state (a case we will cover in part (iv), below); then (28) implies
\[
B(B) = \frac{- (\hat{U}_c - \alpha) c - \alpha g}{\beta \hat{u}_{x\hat{x}}} + \frac{1 + B}{\beta} - 1.
\]
Define $\tilde{B}^1$ such that $B(\tilde{B}^1) = \tilde{B}^0$. Thus, $c = \hat{c}, g = \hat{g}$ and $\tilde{B}^1 = (1 + \tilde{B}^0)(1 + \beta) - 1$. The discontinuity of the equilibrium at $\tilde{B}^0$ carries over at $\tilde{B}^1$, since again the solutions from the left and right differ. We can proceed similarly by defining $B(\tilde{B}^{j+1}) = \tilde{B}^j$ for all $j = 0, \ldots, \infty$, and get
\[
\tilde{B}^j = (1 + \tilde{B}^0) \frac{1 - \beta^{j+1}}{1 - \beta} - 1,
\]
with $C(\tilde{B}^j) = \hat{c}$ and $G(\tilde{B}^j) = \hat{g}$. At all this debt levels, the MPME is discontinuous in $B, C$ and $G$. Given $\tilde{B}^0 < -1$, it is straightforward to verify that $\tilde{B}^j < -1$ for all $j = 0, \ldots, \infty$. 

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A.3.5 Part (iv)

Equation (24) also allows for the possibility of being in steady state. Note that \( \tilde{B}^\infty \equiv \frac{1+\hat{B}_0}{1-\beta} - 1 \) is a steady state. I.e., the government at \( \tilde{B}^\infty \) has enough resources to perfectly smooth distortions over time. Using the expression for \( \hat{B}_0 \), we can write \( \tilde{B}^\infty = \frac{(\tilde{U}_c - \alpha)\tilde{c} - \alpha g}{(1-\beta)u_x x} - 1 \). Given \( \tilde{c} < \hat{c} \) and \( \tilde{g} < \hat{g} \), we verify \( \tilde{B}^\infty > \hat{B} \). At \( \tilde{B}^\infty \) the government is indifferent between implementing \( \{ \hat{x}, \hat{c}, \hat{g} \} \) by staying at \( \tilde{B}^\infty \), and rolling-off the debt as stated in part (v). Since for all \( B \in (\hat{B}, \tilde{B}^\infty) \), the government can implement \( x = x = \hat{x} \), \( c = c' \in (\hat{c}, \hat{c}) \), \( g = g' \in (\hat{g}, \hat{g}) \) forever by keeping a constant debt, it strictly prefers this policy to rolling-off debt. Hence, \( B(B) = B \) for \( B \in [\hat{B}, \tilde{B}^\infty] \).

A.3.6 Part (vi)

We first show \( B(B') > -1 \) for all \( B \geq -1 \). Suppose not. We can rule out \( B' = -1 \) since from part (v), \( B(\hat{B}^0) = -1 \), where \( \hat{B}^0 \in (\hat{B}, -1) \), and by Lemma 2 there is only one such point. Thus, fix some \( B \geq -1 \) such that \( B' < -1 \). Then, \( x \leq \hat{x} \) (strictly if \( B > -1 \)), \( x' = \hat{x} \) and \( \chi''_B = 0 \); thus, (25) implies \( g = g' < \hat{g} \) (by part (ii)) and from (27), \( c = c' < \hat{c} \). From (28) we get \( (U_c - \alpha)c - \alpha g = f_y x (1+B) - \beta f_y x (1+B') > 0 \), given \( B \geq -1 \) and \( B' < -1 \). Let \( K \equiv (U_c - \alpha)c - \alpha g > 0 \). Since \( c = c' \) and \( g = g' \), in the following period, (28) becomes \( K = \tilde{f}_y x (1+B') - \beta \eta x (u_g - f_y') - \beta f_y'' x (1+B'') \). Given \( K > 0 \) and the first two terms of the right-hand side being jointly negative, we get \( B'' < -1 \) and thus, \( x'' = \hat{x} \), \( \chi''_B = 0 \), \( c = c' = c'' \), \( g = g' = g'' \). Thus, (28) simplifies to \( K = \tilde{f}_y x (1+B' - \beta (1+B'')) > 0 \) and so \( B'' < \frac{B'}{\beta} \). Following the same logic, we get debt decreasing in every period, at a rate faster than \( 1/\beta \), i.e., approaching \( -\infty \), while the allocation is \( \{ \hat{x}, c, g \} \) in every period. But for any \( B \leq \hat{B} \) we can implement the first-best, a contradiction with \( c < \hat{c} \), \( g < \hat{g} \).

Now we show \( \chi''_B < 0 \) for all \( B \geq -1 \). Suppose not. From the proof above, we know that \( B' = B(B) > -1 \) for any \( B \geq -1 \). Fix some \( B \geq -1 \) such that \( \chi''_B \geq 0 \). Differentiating (26) with respect to \( B' \) and evaluating it at \( B' \) we get (note \( \zeta' = 0 \) given \( B' > -1 \)): \( (\alpha \eta(u_{xx'} - \eta f_{yy'} 1-\eta)) - (v_y' - \alpha) \eta (2f_{xx'} x + f_{xx'yy'})(1+B') \chi''_B - v_{yy'} (f_y' + f_{yy'}')(1+B') \chi''_B - (v_y' - \alpha)(f_y' + f_{yy'}') = 0 \). Given \( v_y' - \alpha > 0 \), \( \chi''_B \geq 0 \) and Assumption 2, the first and third terms are jointly negative; thus, \( \chi''_B > 0 \). By Lemma 3 we also get \( \zeta''_B > 0 \). Thus, increasing \( B' \) above the prescribed \( B(B) \): (a) (weakly) increases \( x' \) since \( \chi''_B \geq 0 \); (b) increases \( c' \) and \( g' \) since \( \chi''_B > 0 \); and (c) makes the government budget constraint (28) slack by Lemma 4. Thus, the current government can increase welfare by increasing \( B' \), a contradiction with \( \chi''_B \geq 0 \) being an equilibrium.

A.3.7 Part (vii)

Suppose \( \epsilon_{xx'} \chi''_B = 0 \) for some \( B > -1 \). From part (vi), \( \chi''_B < 0 \); thus, \( \epsilon_{xx'} = 0 \), which is an equation in \( B' \). Call the solution \( B^* \), which is a steady state. We get

\[
B^* = -\frac{\eta (u_x^* + u_{xx'} x^*)}{f_y^* + f_{yy'}^* y^*} - 1 + \eta.
\]
We now verify $B^* > -1$. Replace the expression for $B^*$ in (26) and get

$$\eta v^*_y(u^*_x - f^*_y) + (\eta x^*(v^*_y - \alpha) + \alpha \zeta^*) \left( u^*_{xx} - \frac{\eta f^*_y}{1 - \eta} \right) = 0.$$ 

Given $u^*_{xx} - \frac{\eta f^*_y}{1 - \eta} < 0$, the only solution with $v^*_y - \alpha > 0$ features $u^*_x - f^*_y > 0$ and $\zeta^* = 0$. By part (i), $B^* > -1$.

A.4 Proof of Proposition 3

Focus on $B \geq -1$, i.e., when $\zeta = 0$. As $\nu \to 0$, $v_y \to \psi$ and (25) becomes $(\psi - \alpha)\varepsilon x'X_B' = 0$. From Proposition 2, we have $X_B' < 0$ (we verify this property below to ensure it holds in the limit). Thus, since $\psi > \alpha$, $\varepsilon x' = 0$. This is an equation in $B'$ and $X(B')$ only; thus, the solution $B'$ is the same for all $B$, i.e., $B(B)$ is a constant. Given $B(B^*) = B^*$, we get $B(B) = B^*$ for all $B \geq -1$. From (26) we get $\frac{u^*_x - f^*_y}{f^*_y + f^*_{yy}y} = \frac{(\psi - \alpha)(1 + B)}{\alpha \eta}$. Differentiating both sides with respect to $B$ implies

$$\frac{(1 - \eta)u^*_{xx} - \eta f^*_{yy}}{f^*_y + f^*_{yy}y} \cdot \frac{-\eta(v^*_y - f^*_y)(2f^*_y + f^*_{yy}y)}{(f^*_y + f^*_{yy}y)^2} \cdot \frac{X_B'}{1 - \eta} = \frac{\psi - \alpha}{\alpha \eta} > 0.$$ 

Given Assumption 2, the term multiplying $X_B'$ is strictly negative; thus, $X_B' < 0$ for all $B \geq -1$. Applying the same argument as in Proposition 2, part (vii), we have $B^* > -1$; hence, we verify $X'_B = X_0^* < 0$.

A.5 Proof of Proposition 4

Part (i). From (32), $B^* > 0$ implies $-1 - \frac{u^*_x x^*}{u^*_x} > (\frac{1 - \eta}{\eta})(f^*_{yy} + f^*_{yy}y^*)$. The right-hand side of this inequality is positive; thus, $-\frac{u^*_x x^*}{u^*_x} > 1$ is a necessary condition for $B^* > 0$.

Part (ii). As $\eta \to 1$, $B^* \to -\frac{u^*_x + u^*_x x^*}{f^*_y + f^*_{yy}y}$. Given $f^*_y + f^*_{yy}y^* > 0$, $B^* > 0$ iff $u^*_x + u^*_x x^* < 0$.

Part (iii). As $\eta \to 0$, $B^* \to -1$, trivially.

Part (iv). If $u^*_x = x^* - \sigma$ and $f^*_x = \phi$, $B^* > 0$ implies $-1 + \sigma > (\frac{1 - \eta}{\eta})(\frac{\phi}{x^*})$. From Proposition 2, $x^* - \sigma > \phi$; thus, $-1 + \sigma > (\frac{1 - \eta}{\eta})$, i.e., $\eta \sigma > 1$, is a sufficient condition for $B^* > 0$.

A.6 Proof of Proposition 5

The problem with commitment has been analyzed extensively elsewhere\(^\text{16}\), so some steps are skipped for brevity. I will consider the sequence problem of a government with commitment (i.e., the Ramsey problem). A standard result is that the sequence of government budget constraints collapses to a single “implementability” constraint. A simple way to derive it here, is to take (19), multiply it by $\beta^t$ and sum over all periods. Then, use the transversality condition, $\lim_{T \to \infty} \beta^{T(1 + \mu T)(1 + q T B_{T+1})} = 0$, which implies $\lim_{T \to \infty} \beta^{T(U_{c,T} f^*_y y^* x^*_T)} = 0$, where $\{\beta \eta x_{T+1}^*(u^*_{x,T+1} - f^*_y y_{T+1}) + \beta f^*_y y_{T+1} x^*_T (1 + B_{T+1})\} = 0$.

\(^\text{16}\)E.g., see ?, Díaz-Giménez, Giovannetti, Marimón and Teles (2008) and Aruoba and Chugh (2008).
Note that all terms containing debt cancel out, except for the initial period, \( t = 0 \). After some rearrangements, we get

\[ \sum_{t=0}^{\infty} \beta^t \{ (U_{c,t} - \alpha)c_t - \alpha g_t + \eta x_t(u_{x,t} - f_{y,t}) \} - \eta x_0(u_{x,0} - f_{y,0} - f_{y,0}x_0(1 + B_0) = 0. \tag{34} \]

The problem of the government is

\[ \max_{\{x_{t},c_{t},g_{t}\}} \sum_{t=0}^{\infty} \beta^t \{ \eta u(x_t) - (1 - \eta)f(y_t) + U(c_t) - \alpha(c_t + g_t) + v(g_t) \} \]

subject to: (34); \( u_{x,t} - f_{y,t} \geq 0 \) for all \( t \).

Assume \( B_0 > -1 \), so that the non-negativity constraint does not bind in any period (easy to verify). The first-order conditions are

\[
\eta(u_{x,0} - f_{y,0}) - \Lambda(f_{y,0} + f_{y,0}y_0)(1 + B_0) = 0, \quad \text{for } t = 0
\]

\[
(u_{x,t} - f_{y,t}) + \Lambda(u_{x,t} - f_{y,t} + u_{xx,t}x_t - f_{y,y,ty}) = 0, \quad \text{for all } t \geq 1
\]

\[
U_{c,t} - \alpha + \Lambda(U_{c,t} - \alpha + U_{cc,t}c_t) = 0, \quad \text{for all } t \geq 0
\]

\[
v_{g,t} - \alpha - \Lambda \alpha = 0, \quad \text{for all } t \geq 0,
\]

where \( \Lambda \) is the Lagrange multiplier associated with (34). Note that \( c_t \) and \( g_t \) are constant for all \( t \geq 0 \), while \( x_t \) is constant for all \( t \geq 1 \) and may be different in the initial period. Call the corresponding allocation \( \{x_0, x_1, c, g\} \). Thus, we can write (34) as \((U_{c} - \alpha)c - \alpha g + \beta \eta x_1(u_{x,1} - f_{y,1}) = (1 - \beta)f_{y,0}x_0(1 + B_0)\). Plug this expression into the government budget constraint (19) and we get \( B_t = \frac{f_{y,0}x_0(1 + B_0)}{f_{y,0}x_1} - 1 \) for all \( t \geq 1 \), i.e., debt is constant after the initial period as well, which is a standard feature of this type of models. After some rearrangements, \( \{x_0, x_1, c, g\} \) solve

\[
\alpha \eta(u_{x,0} - f_{y,0}) - (v_g - \alpha)(f_{y,0} + f_{y,0}y_0)(1 + B_0) = 0
\]

\[
v_g(u_{x,1} - f_{y,1}) + (v_g - \alpha)(u_{xx,1}x_1 - f_{y,y,1}) = 0
\]

\[
v_g(U_{c} - \alpha) + (v_g - \alpha)U_{cc}c = 0
\]

\[
(U_{c} - \alpha)c - \alpha g + \beta \eta x_1(u_{x,1} - f_{y,1}) - (1 - \beta)f_{y,0}x_0(1 + B_0) = 0.
\]

If we set \( B_0 = B^{*} \), it is straightforward to verify that \( \{x^{*}, x^{*}, c^{*}, g^{*}\} \) solves the above system. Thus, \( B_t = B^{*} \) for all \( t \geq 0 \).

### A.7 Proof of Proposition 6

As \( \rho \rightarrow 1 \) and \( \nu \rightarrow 0 \), we get: \( B^{*} \rightarrow \frac{\eta \sigma}{\psi - \sigma(\psi - \alpha)} - 1; \) \( \mu^{*} \rightarrow \frac{\beta(\psi - \sigma)(\psi - \alpha)(1 - \eta)}{\psi - \sigma(\psi - \alpha)} - 1; \) \( \tau^{*} \rightarrow 1 - \frac{\alpha}{\psi} \); \( g^{*} \rightarrow \frac{1}{\psi \alpha} \left\{ \psi - \sigma + \eta \sigma (\beta \psi - \alpha) \left( \frac{\phi \psi}{\psi - \sigma(\psi - \alpha)} \right)^{1 - \frac{1}{\sigma}} \right\} \). Note that in steady state, inflation is equal to the money growth rate. The restrictions on parameter values imply \( \psi - \sigma(\psi - \alpha) > 0, \beta \psi - \alpha > 0 \) and \( \alpha - (1 - \beta)(\sigma - 1)\psi > 0 \).

Part (i): \( \frac{\partial B^{*}}{\partial \alpha} = \frac{(1 - \sigma)\eta \sigma \psi}{(\psi - \sigma(\psi - \alpha))^{\sigma}}, \) same sign as \( 1 - \sigma; \) \( \frac{\partial \mu^{*}}{\partial \alpha} = -\frac{\partial \eta \sigma \psi}{(\psi - \sigma(\psi - \alpha))^{\sigma}} < 0; \) \( \frac{\partial \tau^{*}}{\partial \alpha} = -\frac{1}{\psi} < 0. \)
Part (ii): \( \frac{\partial B^*}{\partial \psi} = \frac{(\sigma - 1)\eta \sigma}{(\psi - \sigma(\psi - \alpha))^2} \), same sign as \( \sigma - 1 \); \( \frac{\partial \mu^*}{\partial \psi} = \frac{\beta \eta \sigma}{(\psi - \sigma(\psi - \alpha))^2} > 0 \); \( \frac{\partial \tau^*}{\partial \psi} = \frac{\alpha}{\psi^2} > 0 \);

\[ \frac{\partial \eta^*}{\partial \psi} = \frac{1}{\psi^2} + \frac{\eta \psi \sigma(\alpha - (1 - \beta)(\sigma - 1)\psi)(\psi - \sigma(\psi - \alpha))^\frac{1}{2}}{\psi(\phi \psi)^\frac{1}{2}} > 0. \]

Part (iii): \( \frac{\partial B^*}{\partial \sigma} = \frac{\eta \alpha \psi}{(\psi - \sigma(\psi - \alpha))^2} > 0 \); \( \frac{\partial \mu^*}{\partial \sigma} = \frac{\beta \eta \psi(\psi - \alpha)}{(\psi - \sigma(\psi - \alpha))^2} > 0 \); \( \frac{\partial \tau^*}{\partial \sigma} = 0 \);

\[ \frac{\partial \eta^*}{\partial \sigma} = \frac{\eta \phi(\psi - \alpha)}{\alpha \phi(\psi - \alpha)(\psi - \sigma(\psi - \alpha)^{1 - \frac{1}{2}})} \text{, same sign as } \alpha \sigma + (\psi - \sigma(\psi - \alpha)) \ln \frac{\phi \psi}{\psi - \sigma(\psi - \alpha)} \text{, i.e., same sign as } \phi - \kappa \text{, where } \kappa = \psi - \sigma(\psi - \alpha). \]

Part (iv): \( \frac{\partial B^*}{\partial \eta} = \frac{\alpha \sigma}{\psi - \sigma(\psi - \alpha)} > 0 \); \( \frac{\partial \mu^*}{\partial \eta} = \frac{\beta \psi(\psi - \alpha)}{\psi - \sigma(\psi - \alpha)} > 0 \); \( \frac{\partial \tau^*}{\partial \eta} = 0 \); \( \frac{\partial \eta^*}{\partial \eta} = \frac{\sigma - 1}{\alpha \phi \psi} (\frac{\phi \psi}{\psi - \sigma(\psi - \alpha)})^{1 - \frac{1}{2}} > 0 \).

Part (v): \( \frac{\partial B^*}{\partial \phi} = \frac{\partial \mu^*}{\partial \phi} = \frac{\partial \tau^*}{\partial \phi} = 0 \); \( \frac{\partial \eta^*}{\partial \phi} = \frac{\eta(\sigma - 1)(\beta \psi - \alpha)}{\alpha \phi \psi} \left( \frac{\psi - \sigma(\psi - \alpha)}{\psi - \sigma(\psi - \alpha)} \right)^{1 - \frac{1}{2}} \), same sign as \( \sigma - 1 \).
References


