Necessity and Materiality: A New Idiom for the Study of Entailment

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Outline

1 Introduction

2 Hypergraphs

3 Generality of hypergraph lattices

4 Materiality and Necessity
Paraconsistent logics

Explosion

\[ \forall \alpha, \forall \beta : \alpha \land \neg \alpha \vdash \beta. \]
Paraconsistent logics

Explosion

\[ \forall \alpha, \forall \beta : \alpha \land \neg \alpha \vdash \beta. \]

- Dialethic
Explosion

∀α, ∀β : α ∧ ¬α ⊢ β.

- Dialethic
- Preservational
Paraconsistent logics

Explosion

∀α, ∀β : α ∧ ¬α ⊢ β.

- Dialethic
- Preservational
- Algebraic
Hypergraphs

Paraconsistent logics

Explosion

\[ \forall \alpha, \forall \beta : \alpha \land \neg \alpha \vdash \beta. \]

Representational

- Dialectic
- Preservational
- Algebraic
Addenda to the Specimen of the Universal Calculus

All this is easily proved from the one assumption that the subject is as it were a container, and the predicate the simultaneous or conjunctive conten; or conversely, that the subject is as it were a content, and the predicate an alternative or disjunctive container.

UCLA propositions

\[ [\alpha] \]

The set of possible worlds where \( \alpha \) is true.

Inclusion

\[ [\alpha] \subseteq [\beta] \]
Leibniz cont’d

Contingency

When the analysis of a necessary proposition is continued far enough it arrives at an identical equation; that’s what it is to demonstrate a truth with geometrical rigour. But the analysis of a contingent proposition continues to infinity, giving reasons (and reasons for the reasons (and reasons for those reasons...)), so that one never has a complete demonstration. There is always an underlying complete and final reason for the truth of the proposition, but only God completely grasps it, he being the only one who can whip through the infinite series in one stroke of the mind.

Analysis

DNF/CNF

Disjunction of conjunctions of atomic sentences or their negations
Conjunction of disjunctions of atomic sentences or their negations

Subsumption

\[ DNF(\alpha) \sqsubseteq DNF(\beta) \]
\[ CNF(\alpha) \sqsubseteq CNF(\beta) \]
Subsumption between normal forms

\[ \text{CNF}(\alpha) \sqsubseteq \text{CNF}(\beta) \]

\[ \forall C' \in \text{CNF}(\beta), \exists C \in \text{CNF}(\alpha) : C \vdash C'. \]
Hypergraphs

Subsumption between normal forms

\[ \text{CNF}(\alpha) \sqsubseteq \text{CNF}(\beta) \]

\[
\forall C' \in \text{CNF}(\beta), \exists C \in \text{CNF}(\alpha) : C \vdash C'.
\]

Example

\[ p \land q \vdash (p \lor q) \land (r \lor q) \]
Hypergraphs

Subsumption between normal forms

\[ CNF(\alpha) \subseteq CNF(\beta) \]

\[ \forall C' \in CNF(\beta), \exists C \in CNF(\alpha) : C \vdash C'. \]

Example

\[ p \land q \vdash (p \lor q) \land (r \lor q) \]
Hypergraphs

**h-graphs representation**

A hypergraph is a hypergraph on $S$ if and only if $H$ is a non-empty collection of non-empty subsets of $S$.

**Simple hypergraphs/Clutter**

A simple hypergraph is a hypergraph such that there is no proper subset relation between any two edges. (Casting out superedges)
Hypergraphs

h-graphs representation

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Hypergraphs

h-graphs representation

**hypergraphs**

$H$ is a hypergraph on $S$ if and only if $H$ is a non-empty collection of non-empty subsets of $S$.

- Ground set
Hypergraphs

**h-graphs representation**

**hypergraphs**

\[ H \text{ is a hypergraph on } S \text{ if and only if } H \text{ is a non-empty collection of non-empty subsets of } S. \]

- Ground set
- Edges
Hypergraphs

**h-graphs representation**

### hypergraphs

- $H$ is a hypergraph on $S$ if and only if $H$ is a non-empty collection of non-empty subsets of $S$.
  - Ground set
  - Edges
  - Vertices
Hypergraphs

h-graphs representation

**Hypergraphs**

$H$ is a hypergraph on $S$ if and only if $H$ is a non-empty collection of non-empty subsets of $S$.

- Ground set
- Edges
- Vertices

**Simple hypergraphs/Clutter**

A simple hypergraph is a hypergraph such that there is no proper subset relation between any two edges. (Casting out superedges)
Articular Models

An articular model is an ordered triple $\mathcal{M} = \langle U, \mathbb{H}, H \rangle$ where

1. $U \neq \emptyset$ is a set;

2. $\mathbb{H}$ is a set of simple hypergraphs, and to each $p_i$, $H$ assigns a simple hypergraph on $U$, $H(p_i)$.
Articular Models

An articular model is an ordered triple $\mathcal{M} = \langle U, \mathcal{H}, H \rangle$ where

1. $U \neq \emptyset$ is a set;
2. $\mathcal{H} \subseteq \wp(\wp(U))$ such that every member of $\mathcal{H}$ is a simple hypergraph.
Articular Models

An articular model is an ordered triple \( \mathcal{M} = \langle U, \mathcal{H}, H \rangle \) where

1. \( U \neq \emptyset \) is a set;
2. \( \mathcal{H} \subseteq \mathcal{P}(U) \) such that every member of \( \mathcal{H} \) is a simple hypergraph.
3. \( H : At \rightarrow \mathcal{H} \).
Articular Models

An articular model is an ordered triple $\mathcal{M} = \langle U, \mathbb{H}, H \rangle$ where

1. $U \neq \emptyset$ is a set;
2. $\mathbb{H} \subseteq \wp(\wp(U))$ such that every member of $\mathbb{H}$ is a simple hypergraph.
3. $H : At \rightarrow \mathbb{H}$.

That is, $\mathbb{H}$ is a set of simple hypergraphs, and to each $p_i$, $H$ assigns a simple hypergraph on $U$, $H(p_i)$. 
Subsumption

∀H, H′ ∈ H, H ⊆ H′, (H′ subsumes H) if and only if ∀E′ ∈ H′, ∃E ∈ H such that E ⊆ E′.

An Example

r ∧ (p ∨ q) ∧ ¬p ⊢ (p ∨ q) ∧ (¬p ∨ q);

r ∧ (p ∨ q) ∧ ¬p ̸⊢ q.
**Subsumption**

\[ \forall H, H' \in \mathbb{H}, H \subseteq H', (H' \text{ subsumes } H) \text{ if and only if } \forall E' \in H', \exists E \in H \text{ such that } E \subseteq E'. \]

**An Example**

- \( r \land ((p \lor q) \land \neg p) \vdash (p \lor q) \land (\neg p \lor q); \)
- \( r \land ((p \lor q) \land \neg p) \nvdash q. \)
Hypergraphs

Toward hypergraph lattices

Hypergraph operations

- \( H \cap H' = H \cup H' \);
- \( H \uplus H' = \{ a \cup b \mid a \in H, b \in H' \} \).
Hypergraphs

Toward hypergraph lattices

Hypergraph operations

\[ H \cap H' = H \cup H'; \]
\[ H \sqcup H' = \{ a \cup b \mid a \in H, \ b \in H' \}. \]

Lattice operations \( \langle H, \sqsubseteq \rangle \)

\[ H \cap H' = H \land H'; \]
\[ H \sqcup H' = H \lor H'. \]
Hypergraphs

Transverse hypergraph

**intersector**

If $A \subseteq \wp(S)$, then $b$ is an intersector of $A$ iff $\forall a \in A, b \cap a \neq \phi$.

**minimal intersector**

If $A \subseteq \wp(S)$, then $\tau(A) = \{b \mid b$ is a minimal intersector of $A\}$.

**transverse hypergraph/blocker**

If $H$ is a hypergraph, then $\tau(H)$ is the transverse hypergraph of $H$. 
Hypergraphs and simple hypergraphs

\[ \star : H \rightarrow \tau\tau(H) \]

The set of all hypergraphs \( H \) resolves into a set of equivalence classes \([H]\), the hypergraphs in each equivalence class share a unique simple hypergraph.

\[ \star H \subseteq H \subseteq \star H \]
Hypergraphs

Hypergraph lattice

Let $\mathcal{H}$ be the set of all hypergraphs on $\wp(U)$, the lattice $\langle [\mathcal{H}] / \equiv, \leq \rangle$ where

is isomorphic with $\langle \mathcal{H}, \sqsubseteq \rangle$. 
Let $H$ be the set of all hypergraphs on $\mathcal{P}(U)$, the lattice $\langle [H]/\equiv, \leq \rangle$ where

1. $\star H \trianglelefteq H \trianglelefteq \star H$ is an equivalence relation.

is isomorphic with $\langle \mathbb{H}, \sqsubseteq \rangle$. 

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2. $\star$ is a function, $\star(H)$ for every equivalence class is unique.

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Let $\mathbf{H}$ be the set of all hypergraphs on $\mathcal{P}(U)$, the lattice $\langle [\mathbf{H}]/\equiv, \leq \rangle$ where

1. $\star \mathbf{H} \sqsubseteq \mathbf{H} \sqsubseteq \star \mathbf{H}$ is an equivalence relation.
2. $\star$ is a function, $\star(H)$ for every equivalence class is unique.
3. $[\mathbf{H}]_x$ is an equivalence class represented by $x$, $[\mathbf{H}]_x \leq [\mathbf{H}]_y$ if and only if $x \sqsubseteq y$.

is isomorphic with $\langle \mathbb{H}, \sqsubseteq \rangle$. 

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Hypergraphs

**FDE as a hypergraph lattice**

\[(p \lor \neg p) \lor (q \lor \neg q)\]

\[p \lor \neg p \lor \neg q \quad p \lor \neg p \lor q \quad p \lor q \lor \neg q \quad \neg p \lor \neg q \lor q\]

\[p \lor \neg p \quad p \lor \neg q \quad \neg p \lor \neg q \quad p \lor q \quad \neg p \lor q \quad q \lor \neg q\]

\[p \quad \neg p \quad \neg q \quad q\]

\[p \land \neg p \quad p \land \neg q \quad \neg p \land \neg q \quad p \land q \quad \neg p \land q \quad q \land \neg q\]

\[p \land \neg p \land \neg q \quad p \land \neg p \land q \quad p \land q \land \neg q \quad \neg p \land \neg q \land q\]

\[(p \land \neg p) \land (q \land \neg q)\]
Hypergraphs

**FDE as a hypergraph lattice**

\[(p \lor \neg p) \lor (q \lor \neg q)\]

\[p \lor \neg p \lor \neg q\]
\[p \lor \neg p \lor q\]
\[p \lor q \lor \neg q\]
\[\neg p \lor \neg q \lor q\]

\[p \lor \neg p\]
\[p \lor \neg q\]
\[\neg p \lor \neg q\]
\[p \lor q\]
\[\neg p \lor q\]
\[q \lor \neg q\]

\[p \land \neg p\]
\[p \land \neg q\]
\[\neg p \land \neg q\]
\[p \land q\]
\[\neg p \land q\]
\[q \land \neg q\]

\[p \land \neg p\]
\[p \land \neg q\]
\[\neg p \land q\]
\[\neg p \land \neg q\]

\[(p \land \neg p) \land (q \land \neg q)\]
Hypergraphs

**FDE as a hypergraph lattice**

\[(p \lor \neg p) \lor (q \lor \neg q)\]

\[p \lor \neg p \quad p \lor \neg q \quad p \lor q \quad \neg p \lor q \quad \neg p \lor \neg q \lor q\]

\[p \lor \neg p \quad p \lor \neg q \quad \neg p \lor q \quad \neg q \lor \neg q \quad q \lor \neg q\]

\[p \quad \neg p \quad \neg q \quad q\]

\[p \land \neg p \quad p \land \neg q \quad \neg p \land q \quad p \land q \quad \neg p \land q \quad \neg p \land \neg q \lor q\]

\[p \land \neg p \quad p \land \neg q \quad \neg p \land q \quad p \land \neg q \quad \neg p \land \neg q \land q\]

\[(p \land \neg p) \land (q \land \neg q)\]
FDE as a hypergraph lattice

\[(p \lor \neg p) \lor (q \lor \neg q)\]

\[p \lor \neg p\]
\[p \lor q \lor \neg q\]
\[p \lor \neg q \lor q\]

\[p \lor \neg p\]
\[p \lor q\]
\[p \lor \neg q\]

\[p \lor \neg p\]
\[p \lor q\]
\[p \lor \neg q\]

\[p \lor \neg p\]
\[p \lor \neg q\]
\[p \lor q\]

\[p \land \neg p\]
\[p \land \neg q\]
\[p \land q\]
\[p \land \neg q\]

\[p \land \neg p\]
\[p \land \neg q\]
\[p \land q\]

\[(p \land \neg p) \land (q \land \neg q)\]
FDE as a hypergraph lattice
Hypergraphs

FDE as a hypergraph lattice

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Hypergraphs

**FDE as a hypergraph lattice**

(p ∨ ¬p) ∨ (q ∨ ¬q)

Necessity and Materiality: A New Idiom for the Study of Entailment
Hypergraphs

\[
FDE
\]

1. \( \vdash \neg \neg p \leftrightarrow p; \)
2. \( \vdash p \wedge (q \lor r) \rightarrow (p \wedge q) \lor r; \)
3. \( \vdash p \rightarrow p \lor q; \)
4. \( \vdash p \wedge q \rightarrow p. \)

 together with three rules
Hypergraphs

**FDE**

1. \( \vdash \neg
\neg p \iff p \);
2. \( \vdash p \land (q \lor r) \rightarrow (p \land q) \lor r \);
3. \( \vdash p \rightarrow p \lor q \);
4. \( \vdash p \land q \rightarrow p \).

together with three rules

**Transitivity:**

\[
\frac{\vdash \alpha \rightarrow \beta \quad \vdash \beta \rightarrow \gamma}{\vdash \alpha \rightarrow \gamma}
\]
Hypergraphs

\[ FDE \]

1. \( \vdash \neg\neg p \leftrightarrow p; \)
2. \( \vdash p \land (q \lor r) \rightarrow (p \land q) \lor r; \)
3. \( \vdash p \rightarrow p \lor q; \)
4. \( \vdash p \land q \rightarrow p. \)

Together with three rules:

**Transitivity:**

\[
\dfrac{\vdash \alpha \rightarrow \beta \quad \vdash \beta \rightarrow \gamma}{\vdash \alpha \rightarrow \gamma}
\]

**Left disjunctivity:**

\[
\dfrac{\vdash \alpha \rightarrow \gamma \quad \vdash \beta \rightarrow \gamma}{\vdash \alpha \lor \beta \rightarrow \gamma}
\]
Hypergraphs

**FDE**

1. \( \vdash \neg\neg p \leftrightarrow p; \)
2. \( \vdash p \land (q \lor r) \rightarrow (p \land q) \lor r; \)
3. \( \vdash p \rightarrow p \lor q; \)
4. \( \vdash p \land q \rightarrow p. \)

together with three rules

**Transitivity:** \( \vdash \alpha \rightarrow \beta \) \( \vdash \beta \rightarrow \gamma \)
\( \vdash \alpha \rightarrow \gamma \)

**Left disjunction:** \( \vdash \alpha \rightarrow \gamma \) \( \vdash \beta \rightarrow \gamma \)
\( \vdash \alpha \lor \beta \rightarrow \gamma \)

**Contraposition:** \( \vdash \alpha \rightarrow \beta \)
\( \vdash \neg\beta \rightarrow \neg\alpha \)
Generality of hypergraph lattices

Canonical Model

Literal proof-sets

\[ p \implies \]

\[ \neg p \implies \]
Generality of hypergraph lattices

Canonical Model

Literal proof-sets

\[ p \Rightarrow |p| \]
\[ \neg p \Rightarrow |\neg p| \]
The canonical model $\mathcal{M}^*$ is the ordered pair $\langle U^*, V^* \rangle$ where

1. $U^*$: A set of literal proof-sets.
2. $V^*$: $V^*(P_i) = \{\{|P_i|\}\}$. 

Necessity and Materiality: A New Idiom for the Study of Entailment
Generality of hypergraph lattices

Fundamental theorem

**Representation Theorem**

\[ \forall \alpha \in \Phi, \ H^*_\alpha = \{ |\Delta_i| \mid \Delta_i \in \text{CNF}(\alpha), 1 \leq i \leq n \}. \]

For any set \( S \) and any operation \( \uparrow \), we use \( \uparrow[S] \) to denote the set \( \{ \uparrow s \mid s \in S \} \). Given that \( \Delta_i = \{ \delta_i \mid 1 \leq j \leq m_i \} \), \( |\Delta_i| \) denotes the set of proof sets of \( \delta_j \) (\( 1 \leq j \leq m_i \)).
Completeness of FDE in a class of articular models is a corollary of a fundamental theorem for a class of semantic idioms.

**Fundamental Theorem**

$H_\alpha$ and $H_\beta$ are representations of $\alpha$ and $\beta$ in a semantic idiom, $H_\alpha \sqsubseteq H_\beta \iff \alpha \vdash \beta$. 
Materiality and Necessity

Unary property–materiality

Arrow as an operator of material conditional

\[ H_{\alpha \rightarrow \beta} = H_{\neg \alpha} \cup H_{\beta}. \]
Materiality and Necessity

Unary property–materiality

Arrow as an operator of material conditional

\[ H_{\alpha \rightarrow \beta} = H_{\neg \alpha} \cup H_{\beta}. \]

R

\[ W_1 \quad p \rightarrow p; \]
\[ W_2 \quad (p \rightarrow (p \rightarrow q)) \rightarrow (p \rightarrow q); \]
\[ W_3 \quad (p \rightarrow (q \rightarrow r)) \rightarrow (q \rightarrow (p \rightarrow r)); \]
\[ W_4 \quad (p \rightarrow q) \rightarrow (r \rightarrow p) \rightarrow (r \rightarrow q). \]

But,...
Materiality and Necessity

Materiality cont’d

✓

- $p \vdash (q \rightarrow p)$;
- $\neg p \vdash (p \rightarrow q)$.

✗

- $p \land \neg p \vdash q$;
- $p \vdash (q \lor \neg q)$.
Definition

\[ \square \alpha \iff \forall e \in H_\alpha, \exists v \in e \text{ such that } \exists v' \in e : v' = \overline{v}. \]
Materiality and Necessity

FDE+

\[
\begin{align*}
\Box \alpha, \Box \beta & \quad \Rightarrow \quad \Box (\alpha \land \beta) \quad (K^\land) \\
\Box \alpha, \alpha \vdash \beta & \quad \Rightarrow \quad \Box \beta \quad (RM)
\end{align*}
\]

and

\[
\begin{align*}
\alpha \vdash \beta & \quad \Rightarrow \quad \Box (\alpha \to \beta) \quad (RN)
\end{align*}
\]

\[(p \to q \vdash \neg p \lor q)\]
Materiality and Necessity

Binary property

**Theme**

Entailment is a relation.
Entailment is a relation.
Entailment is a relation.

Subsumption relation implicates a hypergraph. Suppose there are $m$ edges of $H_\alpha$ and $n$ edges of $H_\beta$, the representation of $\alpha \rightarrow \beta$ is the set

$$\{\{(E^1_\alpha, E^1_\beta), \ldots, (E^m_\alpha, E^1_\beta)\}, \ldots \{\{(E^1_\alpha, E^j_\beta), \ldots, (E^i_\alpha, E^n_\beta)\}\}$$
Materiality and Necessity

Subsumption in higher order

\[ \alpha \models \beta \iff \forall B \in H_\beta, \exists A \in H_\alpha : \forall a \in A, \exists b \in B \text{ such that } a \leq b. \]

\[ (a, b) \leq (c, d) \iff a \leq c \& b \leq d \]
Future research

- Ascertain the ordering between vertices in the canonical model;
Future research

- Ascertain the ordering between vertices in the canonical model;
- Axiomatization of the higher-order system;
Future research

- Ascertaining the ordering between vertices in the canonical model;
- Axiomatization of the higher-order system;
- The higher-order system preserves entailment as a relation.