Abstract

We uncover a ‘naturally occurring’ first-degree system, AL of Articular Logic that is both relevant and paraconsistent. The principal semantic innovation is an informationally articulated, but nevertheless entirely classical representation of wffs as simple hypergraphs on the power set of a set of possible states. The principal methodological novelty is the general observation that distinct classical representations of wffs can be selected and combined with redeployments of classical inference to accommodate particular inferential requirements such as paraconsistency and relevance.

1 Introduction: the centrality of truth

Paraconsistent studies are in no ideological struggle with classical logic: in fact they offer detailed understandings of proper sublogics of PL that had not hitherto been systematically studied. Labels such as paraconsistentism and classicalism are misapplied. Within paraconsistent studies, what is somewhere referred to as preservationism, and seen as competing with what is called dialetheism might better be understood as a very general proposal for the study of inference, including inference codified by systems previously given dialethic semantic analyses. Preservational studies have themselves

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admitted a multiplicity of methods. It is, for example, historically incor-
rect to characterise preservational logic as ‘the non-aggregative approach
to paraconsistency’. The preservational claims about $\land -$I consist only in
the observation that in a classical setting, $\land -$introduction does not preserve
certain well-defined measures. One of them is coherence level of a set, $\Sigma$ of
sentences, that is, the least $\zeta$ for which there is a $\zeta$ decomposition of $\Sigma$ into
consistent subsets. Another is incoherence dilution of a set, $\Sigma$, of sentences,
that is the least $\zeta$ for which $\Sigma$ includes a $\zeta -$member inconsistent subset.
However, as we shall see in the sequel, measures of level and dilution can
be defined that are indifferent to aggregation. The preservational veto on
$\land -$I is to be contrasted with the dialethic rejection of disjunctive syllogism,
which fails to preserve truth, albeit in a highly specialized semantic setting.

No logical consideration requires us to choose between the preservational
and dialethic approaches. From a philosopher’s point of view the dialethic
approach is the more radical. It requires us to contemplate true contra-
dictions, and not contradictions that reflect the incapacity of a descriptive
language to be both complete and consistent, but contradictions that reflect
the nature of the world. Again the simultaneous satisfaction of contradic-
tory sentences requires a abandonment of the classical understanding of the
connectives. From the same point of view, preservational methods seem
more conservative, since they do not require us to accept true contradic-
tions, nor do they impose re-interpretations of connectives. To repeat, there
is no purely logical grounds for regarding one as correct, and the other as
not. The one says, ‘Suppose that there were true contradictions.’ The other,
‘Suppose a central processor were to receive contradictory data from distinct
sources.’ In each case, an obvious question arises: how are we to draw prin-
cipled inferences in such imagined circumstances?

Nevertheless, there are profound political differences, potentially transform-
ing the role of logic within philosophy. From this larger point of view, it
is the dialethic approach that emerges as the more conservative, and the
preservational approach as the more radical. Each requires a broadening of
the classical point of view. But which presents philosophers with the more
profound, and one might ask, the more salutary adjustment? One could
argue that it is the preservational approach, for the dialethic enshrines the
traditional philosophical centrality of truth in the constitution of logic; the
latter would see it overthrown.

Now philosophers have no better understanding of truth than anyone else.
Indeed the nature of truth has been from very early times a central preoccupation of their discipline. It is among the principal instances of the Socratic paradox. To put the matter plainly, in our acquisition of natural language we acquire a conversational facility with the vocabulary of truth. But since that acquisition does not require that a deeper understanding of alethic vocabulary be accessible, there is no reason to suppose that such a deeper understanding can be achieved. In speaking of truth, there is no need for us to know what we are talking about, nor is there a need for there to be some way for us to come to know what we are talking about. Certainly there is no reason to suppose that talking about truth, the only method that philosophy has so far applied, is itself a reliable method by which to come to know what truth is. Since, as we are often told, philosophy absorbs its own metatheory, all of these observations ought by now to be trite philosophy. So ought this: neither a god nor conversation vouchsafes to us the right vocabulary for understanding. It is the responsibility of the theorist is to choose the language of his theory.

The language of models appropriates the *vocabulary* of truth, but only as a reading for what is independently well defined mathematically. It cannot supply depth, only clarity. Logicians who study inferential preservation in general, do so because they follow Tarski ([9], page 341-375) in being wary of the language of truth.

‘the languages (either the formalized languages or – what is more frequently the case – the portions of everyday language) which are used in scientific discourse do not have to be semantically closed. This is obvious in case of linguistic phenomena and, in particular, semantic notions do not enter in any way into the subject-matter of a science; for in such a case the language of this science does not have to be provided with any semantic terms at all...Semantically closed languages can be dispensed with even in those scientific discussions in which semantic notions are essentially involved.’

Instead such logicians study the capacities of inference systems to preserve mathematically well-defined properties of sets of sentences. Their aim to investigate features of data sets that are preserved classically beneath the supericies of truth-preservation, and then to study systems that preserve those features, even in the absence of truth.
It is obvious that the semantic representation of a sentence is dictated by its composition. This is evident in the case of classical truth-set representations: the truth-set of the negation of $\alpha$ is the complement of the truth-set of $\alpha$; that of a disjunction the union of the truth-sets of its disjuncts, and so on. But truth-sets also destroy compositional information. Thus the truth-set of $p \lor \neg p$ is identical to the truth-set of $q \lor \neg q$ despite the compositional differences between those sentences; similarly for their negations. An inference relation that respects the classical understanding of the connectives cannot both preserve that compositional information and take relations between truth-sets as its semantic currency.

The earliest preservational studies did retain classical truth-sets, and accordingly did not preserve all compositional information about classical theorems and absurdities. It did however preserve compositionally accessible information about the rest. As an example, 3-forcing would permit any inference of the form

$$(\alpha \lor \beta \lor \gamma), (\alpha \lor \beta \lor \delta), (\alpha \lor \gamma \lor \delta), (\beta \lor \gamma \lor \delta)/\alpha \lor \beta \lor \gamma \lor \delta.$$ 

because the set, 

$$\alpha, \beta, \gamma, \alpha, \beta, \delta, \alpha, \gamma, \delta, \beta, \gamma, \delta$$

is 3-harmonic hypergraph of which the premiss set is a particular formulation. It is a rule of $n$-forcing that every $n$-harmonic hypergraph\(^1\) on a set of sentences, formulated as a set of disjunctions of vertices of hyperedges, 3-forces the disjunction of its vertices. In the example, the wff is a direct formulation of a 3-harmonic hypergraph, but the disjunction is also $n$-forced by any set of propositional equivalents sentence. Again, 2-forcing permits any inference of the following form

$$\alpha, \beta, \gamma/(\alpha \lor \beta) \land (\alpha \lor \gamma) \land (\beta \lor \gamma)$$

because the conclusion formulates a 2-uncolourable hypergraph as a disjunction of conjunctions of the vertices of its hyperedges. And again, any wff classically equivalent to that conclusion would be 2-forced by that set of premisses. In general any set, $\Sigma$ of wffs $n$-forces any wff equivalent to the formulation of a 2-uncolourable hypergraph on $\Sigma$. So the question as to whether a wff is $n$-forced by a given set of wffs can be reduced to the question as to whether a hypergraph formulation of it is (a) a hypergraph

\(^1\)Every $n$–tuple of edges has a common vertex.
on the set of premisses, and (b) n-uncolourable.

This is a significant observation, for every propositional wff is equivalent to a conjunction of disjunctions (also to a disjunction of conjunctions). Thus every sentence of propositional logic can be mapped to a hypergraph on its language. But given a model $M = \langle U, V \rangle$ for a propositional language, every hypergraph on a set of sentences of the language can be mapped to a hypergraph on $\varphi(U)$. That is every sentence of propositional language can be represented by a powerset hypergraph. Moreover, for every pair of wffs in non-identical sets of atoms, there is a propositional model in which they are represented by distinct hypergraphs. Such a representation may be described as articular, it can be understood as articulating the information content of the wff in the hyperedges and ultimately in the vertices of a powerset hypergraph. The question as to what constitutes entailment between two wffs has no single answer. It will depend upon what information content we wish to preserve from entailing wff to entailed. In the remainder of this paper we introduce the general methodology of the articular idiom while presenting more specifically an articular analysis of first-degree entailment (FDE), and providing one illustrative variant. Other systems are studied in more detail in [4, 3, 2].

An immediate semantic consequence of this discussion is asserted in the following:

**Principle of Articulation:** Every propositional wff, $\alpha$ has a classical semantic representation as a simple hypergraph $H_\alpha$ on the power set of a set of states.

A logician who has habitually said that for logicians, a proposition is a set, can, with equal propriety, say, ‘A proposition is a simple hypergraph.’ The difference is one of purpose: there is a purpose for which the latter is the better formulation, and the former the worse. From such a point of view the truth-set mentioned in the former is merely the intersection of the union of the edges of the hypergraph.

Accordingly, we can take as the semantic entailment of $\beta$ by $\alpha$ that, for any set of states, the hypergraph $H_\beta$ of $\beta$, subsumes the hypergraph $H_\alpha$ of $\alpha$. That is, every (hyper)edge of $H_\beta$ extends some edge of $H_\alpha$. Intuitively, we can say that $\alpha$ entails $\beta$ iff every informational atom of $\beta$ is classically entailed by some informational atom of $\alpha$. The resulting first-degree impli-
cation we set out here in a binary system, \textit{AL}. \textit{AL} is like preservationist systems in adopting a more discriminating account of what is to be preserved, and represents a redeployment of classical methods, rather than an introduction of semantically new connectives. Even in the simplified setting of propositional logic, the strategy yields a simple sublogic of \textit{PL} that is both relevant and paraconsistent. It bears an apparently close family resemblance to the logic of paradox, \textit{LP} \cite{7}, but, as we shall see, it bears an even closer family resemblance to \textit{FDE} with which it is identical.

2 The systems of articular inference

The language \(L\) of articular inferences has

1. A denumerable set \(A_t\) of atoms \(p_1, p_2, \ldots, p_i, \ldots\);

2. A set \(K\) of logical connectives \(\{\neg, \lor, \land\}\);

3. A set \(\Phi\) of well-formed formulae defined in the usual recursive manner.

In the following, \(\alpha, \beta, \gamma\) and so on are arbitrary well-formed formulae and \(\Gamma, \Sigma,\) and so on are sets of well-formed formulae. The articular logics \(AL\) that arise from the systems of articular inference we will introduce are \textit{binary logics} in the sense of \cite{5}, that is, sets of ordered pairs of the form \(\langle \beta, \alpha \rangle\).

We write \(\beta \vdash \alpha\) if and only if \(\langle \beta, \alpha \rangle \in AL\). For convenience, we write \(\langle \Sigma, \alpha \rangle \in AL (\Sigma \vdash \alpha)\) as an abbreviation of \(\exists \beta_1, \beta_2, \ldots, \beta_n \in \Sigma\) such that \(\langle \beta_1 \land \beta_2 \ldots \land \beta_n, \alpha \rangle \in AL\).

\(AL\) has two variants, \(AL_1\) and \(AL_2\).

3 Articular semantics and \(AL_1\)

An articulate model is a triple \(M = \langle U, \mathcal{H}, \mathcal{H} \rangle\) where

1. \(U \neq \emptyset\) is a set;

2. \(\mathcal{H} \subseteq \phi \phi(U)\);

3. \(\mathcal{H} : \ At \rightarrow \{H \mid H \in \mathcal{H} \& H \text{ is a simple hypergraph}\}\).

That is, to each \(p_i, \mathcal{H}\) assigns a simple hypergraph on \(\varphi(U)\), denoted by \(H(p_i)\).
Definition 1. A simple hypergraph $H = \{E_1, E_2, \ldots, E_n\}$ is a hypergraph such that if $\forall E_i, E_j \in H, E_i \not\subset E_j$.

The account of $H$, which extends $H$ to $\Phi$, requires some preliminary definitions.

Definition 2. If $A \subseteq \wp(U)$, then $b$ is an intersector of $A$ iff $\forall a \in A, b \cap a \neq \phi$.

Definition 3. If $A \subseteq \wp(U)$, then $\tau(A) = \{b \mid b$ is a minimal intersector of $A\}$.

Definition 4. If $A \subseteq \wp(U)$, then $[A] = \{a \mid a \in A\}$.

Simplification Not all of the set-theoretic operations introduced below preserve simplicity. However, for present purposes, every non-simple hypergraph $H$ has a simple equivalent, $\ast H$, which is obtained by casting out super-edges.

Definition 5. Let $H(U)$ be the set of hypergraphs on $\wp(U)$. Then $\forall H \in H(U), \ast H$ is $H - (E \in H \mid \exists E' \in H : E' \subset E)$.

Definition 6. $H \sqcup H' = \ast\{\{a \cup b\} \mid a \in H, b \in H'\}$
$H \cap H' = \ast\{a \mid a \in H \text{ or } a \in H'\}$
$\overline{H} = H([B_i] \mid B_i \in \tau(H))$.

$H$ extends $H$ to $\Phi$ as follows:
$H_{P_i} = H(P_i)$
$H_{\neg \alpha} = \overline{H}$
$H_{\alpha \lor \beta} = H_{\alpha} \sqcup H_{\beta}$
$H_{\alpha \land \beta} = H_{\alpha} \cap H_{\beta}$.

Definition 7. $\forall H, H' \in H(U), H \sqsubseteq H'$, ($H$ is subsumed by $H'$) iff $\forall b \in H'$, $\exists a \in H$ such that $a \subseteq b$ and $\forall a' \in H$, $\exists b' \in H'$ such that $a' \subseteq b'$.

Definition 8. $\forall \alpha, \beta, \alpha \vdash \beta$ ($\alpha$ entails $\beta$) iff $\forall M = \langle U, H, H \rangle, H_{\alpha} \sqsubseteq H_{\beta}$.
Alternatively, we say that $\alpha \vdash \beta$ is valid. So, mutatis mutandis, for $\Gamma \vdash \alpha$.

Lemma 1. $\langle \ast[H(U)], \sqsubseteq \rangle$ is a lattice.

Proof. It is easily seen that $\sqsubseteq$ is a partial ordering and that
$\forall \alpha, \beta \in \Phi, \sup(H_{\alpha}, H_{\beta}) = H_{\alpha \lor \beta}$ and
$\forall \alpha, \beta \in \Phi, \inf(H_{\alpha}, H_{\beta}) = H_{\alpha \land \beta}$. 

2By $\overline{a}$ we mean the set theoretic complement of $a$. 


In representing the rules and axioms that \( \vdash_1 \) picks up, we use \( \vdash \) to refer to \( \vdash_1 \). It is straightforward to verify that the \( \vdash_1 \) of \( AL_1 \) satisfies the following rules:

**Cut** \( \Gamma, \alpha \vdash \beta, \Gamma \vdash \alpha / \Gamma \vdash \beta \).

In addition, \( \vdash \) satisfies the following rule governing conjunction:

**RC** \( \Gamma \vdash \alpha, \Gamma \vdash \beta / \Gamma \vdash \alpha \land \beta; \) (right conjunctivity)

\( AL_1 \) has the following eight binary axioms.

1. \( \alpha \vdash \alpha \);
2. \( \neg(\alpha \land \beta) \vdash \neg\alpha \lor \neg\beta \);
3. \( \neg(\alpha \lor \beta) \vdash \neg\alpha \land \neg\beta \);
4. \( \neg\neg\alpha \vdash \alpha \);
5. \( \alpha \land \beta \vdash \beta \land \alpha \);
6. \( \alpha \lor \beta \vdash \beta \lor \alpha \);
7. \( \alpha \lor (\beta \land \gamma) \vdash (\alpha \lor \beta) \land (\alpha \lor \gamma) \);
8. \( \alpha \land (\beta \lor \gamma) \vdash (\alpha \land \beta) \lor (\alpha \land \gamma) \).

### 4 \( AL_2 \)

\( AL_2 \) can be obtained from \( AL_1 \) by relaxing the subsumption relation in way of the following definition

**Definition 9.** \( \forall H, H' \in H(U), H \subseteq H', (H \text{ is subsumed by } H') \text{ iff } \forall b \in H', \exists a \in H \text{ such that } a \subseteq b \).

It can be verified that \( AL_2 \) has all the rules that \( AL_1 \) has, in addition to the following:

**Mon** \( \alpha \vdash_2 \beta / \Gamma, \alpha \vdash_2 \beta \);

**Ref** \( \alpha \in \Gamma / \Gamma \vdash_2 \alpha \).

and that therefore \( \vdash_2 \) is a consequence relation in the sense of Scott [8]. In addition to [RC], \( \vdash_2 \) also satisfies the following rule governing disjunction:

\(^3\)The subscript suggests that the binary axioms belong to \( AL_1 \).
Apart from the nine binary axioms for $AL_1$, $AL_2$ has two other axioms.

1. $\alpha \land \beta \vdash \beta$;
2. $\alpha \vdash \beta / \neg \beta \vdash \neg \alpha$.

It is easily checked that the rules [LD] and [RC] preserve validity, and that the listed binary axioms are valid.

5 Metatheory of $AL$

**Metatheorem 1.** *Both $AL_1$ and $AL_2$ are sound with respect to a class of articulate models.*

The first axiom for $AL_1$ becomes a particular instance of [Mon], therefore can be omitted for $AL_2$.

The proof of completeness of $AL$ requires some preliminary definitions.

**Definition 10.** $\Sigma$ is a complete set of literals iff $\Sigma \subseteq At \cup \neg [At]$ and $\forall p_i \in At, p_i \in \Sigma \iff \neg p_i \notin \Sigma$.

**Definition 11.** $\Sigma$ is an $AL$ full theory if and only if it is the $\vdash$-closure of a complete set of literals.

Notice that the set of $AL$ full theories is identical to the set of maximal PL-consistent sets. $|\alpha|$ (the proof set of $\alpha$) is the set of $AL$ full theories that contain $\alpha$.

The $AL$-canonical model $\mathcal{M}^*$ is the ordered pair $\langle U^*, H^* \rangle$ where

1. $U^*$ is the set of $AL$ full theories;
2. $H^*$ is defined piecewise by: $H^*(p_i) = \{|p_i|\}$.

The burden of proof for the fundamental theorem lies in the generalization of $H^*$ to every wff of $AL$, so that each wff is assigned a corresponding hypergraph on the power set of $AL$ full theories.
Definition 12. ∀α ∈ Φ, A(α) is the family of proof-sets of literals, disjoined in the conjuncts of the CNF of α. Thus, where $CNF(\alpha) = \bigwedge_{i=1}^{n} \Delta_i$ and

\[ \Delta_i = \bigvee_{i=1}^{m} \delta_i, \]

then ∀α ∈ Φ, \(H^*_\alpha = \{\{|\delta^1|, |\delta^2|, \ldots, |\delta^n|\} \mid \delta_i \in \Delta_i \land \Delta_i \in A(\alpha)\} \).

It is easily shown that

Metatheorem 2. \(H^*_\alpha =_{def.} A(\alpha)\).

Definition 13. ∀α ∈ Φ, \(L(\alpha)\) (the language of \(\alpha\)) is the set of atoms having occurrences in \(\alpha\).

Metatheorem 3. If \(L(\alpha) = \{p_1, p_2, \ldots, p_n\}\), then \(\alpha \vdash \bigwedge_{i=1}^{n} \Delta_i\) where

\[ \Delta_i = \bigvee_{i=1}^{m} \delta_i, \quad L(\bigwedge_{i=1}^{n} \Delta_i) = L(\alpha), \quad \text{and} \quad \delta_i \text{ is either } p_j \text{ or } \neg p_j (1 \leq j \leq n).\]

Metatheorem 4. \(\alpha \models \beta \Rightarrow \alpha \models \beta\)

AL is complete with respect to the class of articulate models.

Metatheorem 5. AL is decidable.

The demonstration, which we omit here, requires only standard classical filtration methods, since failure of AL-provability is a failure of a classical provability. If we define a restriction of the AL-canonical universe to the proof set of the set of premisses, the unprovable wff will fail in the filtration of that submodel through that wff.

Metatheorem 6. AL is first-degree entailment.

In [1], the first-degree entailment \(A \rightarrow B\) is valid if and only if ∀a ∈ \(\tau(A(A))\), and ∀b ∈ \(A(B)\), \(a \rightarrow b\) is explicitly tautological, i.e. a and b share a common variable. Such is the case in the canonical model for AL. In the ordinary models of AL, the semantic representation of first-degree entailment is ∀a ∈ \(\tau(H)\), and ∀b ∈ \(H'\), \(a \cap b \neq \emptyset\). It is easily demonstrated that this is equivalent with \(H \subseteq H'\).
6 Some concluding remarks about $AL$

The theorems of $AL$ and the validities are binary; the notion of a tautology as a wff entailed by the empty set is inimical to the idea of articular inference, as perhaps it ought to be inimical to relevant inference more generally. Since the $\to$ of $AL$ is defined as $\neg \alpha \lor \beta$, $AL$ does not distinguish between *modus ponens* and *disjunctive syllogism*. Both fail in $AL$.

Although our semantic account of inference relies wholly upon hypergraphs, truth sets, though derivative, are not wholly absent. They are obtained by melting articulations into single sets.

$$[[\alpha]]^M = \bigcap_{i=1}^{n} \{ \bigcup f_i \mid f_i \in H_\alpha, 1 \leq i \leq n \}$$

Thus, if $x \in [[\alpha]]^M$ & $\alpha \models \beta$, then $x \in [[\beta]]^M$. Modus Ponens is a rule of inference of $AL$. That is, $\models$ preserves satisfaction.

Evidently the subsumption lattice imposes strict variable-sharing requirements upon articular inference. We would maintain that they are the intuitively correct requirements. At any rate, they are sufficient to distinguish inferentially both pairs of classical tautologies and pairs of classical inconsistencies that share no variables. Thus, for example,

$$p \lor \neg p \not\models q \lor \neg q$$
$$p \land \neg p \not\models q \land \neg q$$

And of course, $AL$-inference is non-explosive.

$$p \land \neg p \not\models q$$

Not surprisingly, the inference relation $\vdash$ for $AL$ is sensitive to the definition of the subsumption relation. As an example, we can adopt the stronger subsumption relation of [6], which requires that every hyperedge of the subsumed hypergraph be a subedge of some edge of the subsuming hypergraph. Such an adoption will validate a rule of transposition for $\vdash$.

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References


