Articular Models for First-degree Paraconsistent Systems *

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Abstract

We present a class of models (h-models) in which the sentences of a propositional language are represented as simple hypergraphs on the power set of a universe, and in which entailments are modeled by relations between hypergraphs. An h-model theoretic semantic analysis of the system FDE of first-degree entailment is proposed and a non-constructive completeness proof given in the familiar idiom of Henkin models. Finally we introduce a hitherto unstudied system of analytic entailment and provide its natural h-model representation.

1 Introduction

Paraconsistent inference systems are characterized by the absence of the explosion rule

\{\alpha, \neg\alpha\} \vdash Q

Theorists who wish to contrive subsystems of PL that lack the principle must either advance reasons that render their residual systems reasonable or at least define models that reject the principle. In the latter case they have the

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further double burden of rendering their models plausible and justifying as plausible the principles that their models admit. On the former side of this divide we can place Parry (in Fine's axiomatisation [5]) whose prescriptive principle [6] was that no atom should appear on the right side of an arrow that did not appear on the left. He later adduced reasons why that principle alone did not define the analytic implication that he was after. In particular he would admit no principle that ‘demodalised’ implication, as did Dunn’s suggested principle (in [4])

\[ \text{JMD } p \rightarrow ((p \rightarrow p) \rightarrow p) \]

In Parry’s company could be lodged Anderson and Belnap whose system of tautological entailment [1] stipulated as a requirement for \( \alpha \vdash \beta \), that some conjunct of the CNF of \( \alpha \) classically yield some disjunct of the DNF of \( \beta \). On the other side of that divide we can place dialethists such as Priest who propose models that permit the simultaneous satisfaction of \( \alpha \) and \( \neg \alpha \). With Priest can dwell preservationists such as Schotch and Jennings whose models distinguished the set \( \{ \alpha \land \neg \alpha \} \) from the set \( \{ \alpha, \neg \alpha \} \), disarming the latter by partitioning into consistent subsets but permitting unpartitionable \( \{ \alpha \land \neg \alpha \} \) to explode. Both dialethists and preservationists bear the double burden of plausibility. The latter must make a case that the inferential power of Dion is walking; Dion is chewing gum differs significantly from that of Dion is walking and Dion is chewing gum. The former must make a case for models that permit the truth of Dion is walking and Dion is standing still. These are of course caricatures, for on the one hand we might obtain sentences from different sources and their conjunction from none, and on the other, the apparently puzzling claim about Dion, one can argue, ought to be distinguished from the correspondingly puzzling claim Plato is walking and Plato is standing still.

It is the classical logician who must convince us that the claim about Plato follows from the claim about Dion. Nevertheless one can sympathise, contra the preservationists, with the requirement that the members of the second pair should have distinct representations, and, again contra the preservationists, fail in the requirement that the first pair should not. One wants to ask ‘Why not merely construe the notion of level in such a way that the last two conjunctions are of level two?’ What should it matter that the level of a conjunction should be the level of the set of its conjuncts. After all, we are often enough beset with false mathematical beliefs about, say, what follows from what. Such beliefs arise from some single person’s misconception: there is no need to compartmentalise one’s intellect as Davidson
recommended [3] rather than merely face the fact of one’s own mathematical inadequacy. Such beliefs amount to outright contradictions, yet we need not accept that any mathematical error commits us to every mathematical error, including those from which we have been cured.

A representation that distinguishes the conjunction \{\alpha \land \lnot \alpha\} from the conjunction \{\beta \land \lnot \beta\} if \alpha is independent of \beta, but does not distinguish the set \{\alpha \land \lnot \alpha\} from the set \{\alpha, \lnot \alpha\} need be neither dialethic nor non-preservationist. If the correctness of an inference from \alpha to \beta depends upon a defined relationship between the representation of \alpha in a model and the representation of \beta in that model, then there is certainly hope for a first-degree, level-preserving implicational system, provided that in such models the representation of conjunction itself preserves the required distinctions. Evidently such models cannot represent conjunction as an intersection of truth-sets of conjuncts. How they can be represented is the main innovation of this paper.

Such a representation is not to be found in classical semantic practice, where formulae are represented by truth-sets; among the wffs are all those of the form \alpha \land \lnot \alpha that cannot but be represented as the same set, namely, the empty set. If we are to preserve the distinctions between different contradictions, that of \(p \land \lnot p\) as opposed to \(q \land \lnot q\), for example, whilst retaining our familiar understandings of negation and conjunction, then we are bound to find recognizably classical but not merely truth-set-theoretic representations for them. From this point of view, truth-sets are semantic black holes into which distinctions are drawn, and by which they are inexorably and irretrievably obliterated. A semantic idiom that seeks to reject inferences in the light of these distinctions must anchor its representations a safe distance from the event horizons of truth-sets.

Now the universes of classical propositional models do offer such desirable locations: we have merely to find a suitable language in which to stake a semantic claim to them. The germ of the solution is already present, though in a highly specific, constructive application, in Anderson and Belnap’s work. It lies in the simple fact that every propositional formula has both an equivalent conjunctive normal form (CNF) and an equivalent disjunctive normal form (DNF). In what follows we exploit the former, though our idiom could as readily exploit the latter. To preserve compositional differences, as, for example between \(p \land \lnot p\) and \(q \land \lnot q\) into semantics, the goal of this paper, requires us to keep the structure of these formulae in our semantic repre-
sentation. To this end, as in [2], we represent propositional formulae as simple hypergraphs on the power set of the universe. The location of the hypergraphs reveals a link between our hypergraph semantics and classical truth-set semantics, namely that literals of a wff retain a set representation as vertices of hyperedges. To this classical hypergraph representation we now turn.

2 The central historical case

We shall present two systems of first-degree entailments and their hypergraph semantics. As we shall see, the idiom brings one well-known system within the purview of standard Henkin-type(-type) methods. The second system serves to illustrate that the hypergraphic idiom opens up explorations of hitherto unstudied systems.

The preliminaries are the usual. The language of first-degree entailment (in BNF) is defined by

\[
\alpha ::= p | \neg \alpha | (\alpha \lor \alpha)
\]

where p ranges over \(At\), the set of atomic well-formed formulae. In the following, \(\alpha, \beta, \gamma\) are arbitrary well-formed formulae and \(\Gamma, \Sigma\), and so on are ensembles of well-formed formulae. The systems of inference we will introduce are first-degree systems, that is, their underlying language has no nested entailments. In fact, in this presentation we abandon the \(\rightarrow\) in favour of \(\vdash\). The reason is that the former is a connective; the latter, like entailment and implication, is a relation, which we interpret semantically as a relation between hypergraphs. (Let the notorious Use-Mention Committee take note.) The first system, FDE, has been given various semantic treatments. This is the system presented by Anderson and Belnap in [1].

The principles of the system are as follows:

1. \(\neg(\alpha \land \beta) \vdash \neg \alpha \lor \neg \beta\);
2. \(\neg(\alpha \lor \beta) \vdash \neg \alpha \land \neg \beta\);
3. \(\neg \neg \alpha \vdash \alpha\);
4. \(\alpha \land \beta \vdash \beta \land \alpha\);
5. $\alpha \lor \beta \vdash \beta \lor \alpha$;
6. $\alpha \lor (\beta \land \gamma) \not\vdash (\alpha \lor \beta) \land (\alpha \lor \gamma)$;
7. $\alpha \land (\beta \lor \gamma) \vdash (\alpha \land \beta) \lor (\alpha \land \gamma)$;
8. $\alpha \vdash \alpha \lor \beta$.

The system satisfies the central three structural rules:

<table>
<thead>
<tr>
<th>Rule</th>
<th>Definition</th>
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</thead>
<tbody>
<tr>
<td>Mon</td>
<td>$\alpha \vdash \beta / \Gamma$, $\alpha \vdash \beta$;</td>
</tr>
<tr>
<td>Ref</td>
<td>$\alpha \in \Gamma / \Gamma \vdash \alpha$;</td>
</tr>
<tr>
<td>Cut</td>
<td>$\Gamma, \alpha \vdash \beta$, $\Gamma \vdash \alpha / \Gamma \vdash \beta$.</td>
</tr>
</tbody>
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together with

<table>
<thead>
<tr>
<th>Rule</th>
<th>Definition</th>
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<tbody>
<tr>
<td>LD</td>
<td>$\beta \vdash \alpha$, $\gamma \vdash \alpha / \beta \lor \gamma \vdash \alpha$; (left disjunctivity)</td>
</tr>
<tr>
<td>RC</td>
<td>$\Gamma \vdash \alpha$, $\Gamma \vdash \beta / \Gamma \vdash \alpha \land \beta$. (right conjunctivity)</td>
</tr>
</tbody>
</table>

### 3 Hypergraph Semantics

A hypergraph model (h-model) as defined in [2] is a triple $M = \langle S, H, \mathcal{H} \rangle$ where

1. $S \neq \emptyset$ is a set;
2. $H \subseteq \mathcal{P}(S)$;
3. $\mathcal{H} : At \rightarrow \{H \mid H \in H \land H$ is a simple hypergraph$\}$.

That is, to each $p_i$, $\mathcal{H}$ assigns a simple hypergraph on $\wp(S)$, $H(p_i)$.

**Definition 1.** A simple hypergraph $H = \{E_1, E_2, \ldots, E_n\}$ is a hypergraph such that if $\forall E_i, E_j \in H$, $E_i \not\subset E_j$.

The account of $H(\ast)$, which extends $\mathcal{H}$ to $\Phi$, requires some preliminary definitions.

**Definition 2.** If $A \subseteq \wp(S)$, then $b$ is an intersector of $A$ iff $\forall a \in A, b \cap a \neq \emptyset$.

**Definition 3.** If $A \subseteq \wp(S)$, then $\tau(A) = \{b \mid b$ is a minimal intersector of $A\}$. 
**Definition 4.** If $A \subseteq \wp(S)$, then $[A] = \{a \mid a \in A\}$.

**Simplification** Not all of the set-theoretic operations introduced below preserve simplicity of hypergraphs. However, for present purposes, every non-simple hypergraph $H$ has an equivalent simple hypergraph, $\star H$, which is obtained by casting out super-edges. Our attention is focused upon simple hypergraphs.

**Definition 5.** $\forall H \in \mathbf{H}$, $\star H$ is $H - \{E \mid \exists E' \in H : E' \subset E\}$.

$H(_\cdot)$ extends $\mathbf{H}$ to $\Phi$ as follows:

- $H_P = H(P_i)$
- $H_{\neg \alpha} = \{[B_i] \mid B_i \in \tau(H_{\alpha})\}$
- $H_{\alpha \lor \beta} = \star \{\{a \cup b\} \mid a \in H_\alpha, b \in H_\beta\}$
- $H_{\alpha \land \beta} = \star (H_\alpha \cup H_\beta)$.

**Definition 6.** $\forall H, H' \in \mathbf{H}, H \sqsubseteq H'$, (H is subsumed by $H'$) iff $\forall b \in H'$, $\exists a \in H$ such that $a \subseteq b$.

**Definition 7.** $\forall \alpha, \beta$, $\alpha \vdash \beta$ ($\alpha$ semantically entails $\beta$) iff $\forall M = \langle U, H \rangle$, $H_\alpha \sqsubseteq H_\beta$. Alternatively, we say that $\alpha \vdash \beta$ is valid. So, mutatis mutandis, for $\Gamma \vdash \alpha$.

**Lemma 1.** $\langle \star [H], \sqsubseteq \rangle$ is a lattice.

**Proof.** It is easily seen that $\sqsubseteq$ is a partial ordering and that

- $\forall \alpha, \beta \in \Phi$, $\sup(H_\alpha, H_\beta) = H_{\alpha \lor \beta}$
- $\forall \alpha, \beta \in \Phi$, $\inf(H_\alpha, H_\beta) = H_{\alpha \land \beta}$.

The lattice is represented in figure ??.

The lattice accommodates representations of two independent variables, the least base capable of illustrating the inferential distinctiveness of the system.

## 4 Completeness of FDE

**Definition 8.** $\Sigma$ is a complete set of literals iff $\Sigma \subseteq At \cup \neg[At]$ and $\forall p_i \in At$, $p_i \in \Sigma \iff \neg p_i \notin \Sigma$.

**Definition 9.** $\Sigma$ is an $h$ full theory if and only if it is the $\vdash$-closure of a complete set of literals.
Figure 1: FDE
Notice that the set of full theories is identical to the set of maximal PL-consistent sets. \(|\alpha|\) (the proof set of \(\alpha\)) is the set of full theories that contain \(\alpha\).

Denote FDE by \(L\). The \(l\)-canonical h-model \(M_L\) is the ordered triple \(\langle U_L, H_L, H'_L \rangle\) where

1. \(U_L\) is the set of full theories;
2. \(H_L = \{\{|P_i|\} | P_i \in At\}\)
3. \(H'_L\) is defined piecewise by: \(V_L(P_i) = \{|P_i|\}\).

**Definition 10.** The language of \(\alpha\), denoted by \(L(\alpha)\), is the set of atoms occurring in \(\alpha\).

**Definition 11.** \(CNF(\alpha) = \bigwedge_{i=1}^{n}[\bigvee_{k=1}^{m} P_k]_i\)

**Lemma 2.** \(\alpha \models CNF(\alpha)\).

Given the axioms and rules of FDE, we can prove

\(\alpha \models_{\mathcal{M}_L} \beta \text{ iff } CNF(\alpha) \vdash CNF(\beta)\)

therefore by lemma 2,

\(\alpha \models \beta \text{ iff } \alpha \vdash \beta\)

Hence FDE is determined by the class of h-models where \(\alpha \models \beta \text{ iff } H \subseteq H'\).

## 5 H-models generalized

The h-graph semantic idiom introduced in the last section is specific to Boolean algebras. \(H\) in the class of h-models introduced above is a set of hypergraphs on the subset algebra \(b(S)\) of a set \(S\). If our only interest were in bringing FDE within the purview of standard semantic techniques, that idiom would be sufficient. But the interest in the idiom would be sorely restricted, and, as Colonel Fitzwilliam might say, the glory of the achievement sadly diminished. For the discovery and study of other first-degree systems, and for extending the application of the technique to higher-degree systems, greater generality is required. A moment’s (perhaps a few moments’) reflection will reveal that even the semantic analysis of FDE requires no such particularity. The locus of generalisation is in the definition of subsumption,
which we here rehearse.

\( \forall H, H' \in H, H \subseteq H' \), \((H \text{ is subsumed by } H')\) iff \( \forall b \in H', \exists a \in H \) such that \( a \subseteq b \).

First we should note that since the hypergraphs of \( H \) are powerset hypergraphs, the edges of these hypergraphs are themselves hypergraphs on the underlying set. Thus the subset relation holding between an edge \( E \) of the subsumed and \( E' \) the subsuming hypergraphs is actually a subgraph relation requiring the following:

\[ \forall v \in E, \exists v' \in E': v = v'. \]

The generalisation replaces the identity with an inequality:

\[ \forall v \in E, \exists v' \in E': v \leq v'. \]

thereby generalizing the relation between edges:

**Definition 12.** \( E \preceq E' \iff \forall v \in E, \exists v' \in E': v \leq v' \),

which yields a new, generalised relation of subsumption*:

**Definition 13.** \( \forall H, H' \in H, H \subseteq^* H' \iff \forall B \in H', \exists A \in H \) such that \( \forall a \in A, \exists b \in B \) and \( a \preceq b \).

An h-model, on this more general understanding, is a triple \( M = \langle S, H, H \rangle \) where

1. \( S \neq \emptyset \) is a set;
2. \( H \) is a class of hypergraphs on an algebra \( A(S) \) generated by a partial order on the set \( S \);
3. \( H : At \rightarrow \{H \mid H \in H \& H \text{ is simple} \} \).

That is, to each \( P_i \), \( H \) assigns a simple hypergraph, \( H(P_i) \), on an algebra generated by a partial order on the set \( S \). \( \preceq \) is a partial order generated by the basic partial order \( \subseteq \) for the class of general h-models.

The guide to the generalisation will have reminded the canny reader of the close connection between the more particular idiom and the obvious syntactic characterization of Anderson and Belnap’s original tautological entailment. In the first place, such a syntactic account would be equivalent to the
one by which $\alpha$ proves $\beta$ iff every conjunct of the CNF of $\beta$ is classically proved by some conjunct of the CNF of $\alpha$. Since those conjuncts would be disjunctions of literals, the constituent provabilities would require sub-disjunctions, which would require that every disjunct of the one conjunct be identical to some disjunct of the other. In the general case, all that is required is that every disjunct of the one yield some disjunct of the other in the underlying system.

Obviously, $\star[H]$ generates a distinct class of lattices for FDE. FDE is determined by both classes of $h$-models. We have demonstrated completeness for the class of general $h$-models, for as we can see, the subsumption relation in the earlier class of $h$-models is a special case of the subsumption relation defined for the class of general $h$-models.

6 FDAE

The subsumption* relation $\sqsubseteq^*$ excites hope for prescriptive principles that can be directly defined on hypergraphs, such as for example that the set $V(H)$ of vertices of one hypergraph bear some relation, $R$ to the set $V(H')$ of another:

$$V(H_\beta) R V(H_\alpha)$$

In what follows, we consider as a prescriptive principle:

[A] $R$ is inclusion.

The system that is generated by the subsumption relation in the class of general $h$-models along with the prescriptive principle is FDE without the following principle:

$$\alpha \vdash \alpha \lor \beta$$

We call the system FDAE, with A for ‘analyticity’. The system differs from that of Parry (in Fine’s axiomatisation), in notable ways, first in not having the principle:

$$\alpha \lor (\beta \land \neg \beta) \vdash \alpha.$$ 

and second in being based upon a system of entailment rather than upon a system ($S4$) of strict implication.
FDAE is sound and complete with respect to the class of general h-models in which entailment relation is represented by the subsumption relation “$\sqsubseteq^*$” under the constraint of [A]. The smallest lattice representing $*$[$H^*$] is given below. Again it requires representations of at least two independent variables in the lattice as represented in figure 2 to illustrate the inferential distinctiveness of the system.

References


