Some Spectral Facts

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Definitions

A good introduction to Spectral Graph Theory is found in "Spectra of Graphs" by Cvetkovic, Doob and Sachs (CDS, 1980).

Assume G is an undirected connected loopless graph without multiple edges which is not complete. (The definitions below can be extended to weighted graphs, but for simplicity we will not consider multigraphs here. Assuming G is connected and not complete avoids certain trivial results). G has nodes V and edges E, with $|V| = n$.

The *Adjacency Matrix* A(G) of graph G is a binary matrix with

$A(i,j) = 1$ if i is connected to j

$= 0$ otherwise

The eigenpairs of A are (α_i, a_i) such that $Aa_i = \alpha_i a_i$

The eigenvalues $\{\alpha\}$ are called the *spectrum* of A. The eigenvectors a_i are orthogonal.

If G is k-regular, then $a_0 = 1/\sqrt{n}$, with $\alpha_0 = \max(\alpha_i) = k$.

The *Laplacian Matrix* L(G) is a matrix with

$L(i,j) = -1$ if i is connected to j

$L(i,i) = \deg(i)$ where deg(i) is the degree of node i

$L(i,j) = 0$ otherwise

The eigenpairs of L are (λ_i, l_i) with $\lambda_0 = \min(\lambda_i) = 0$ and $l_1 = 1/\sqrt{n}$

The l_i are mutually orthogonal and the Laplacian spectrum is $0=\lambda_0 \leq \lambda_1 \leq ,..., \leq \lambda_{n-1} \leq n$

The multiplicity of 0 as an eigenvalue is equal to the number of components in G.

There are a number of other equivalent definitions of L the simplest being:

$L = D - A$

where D is the diagonal matrix of node degrees.

The *Normal matrix* N(G) is a matrix with

$N(i,j) = 1/(\sqrt{\deg(i)}\sqrt{\deg(j)})$ if i is connected to j

$N(i,j) = 0$ otherwise

We may view N as $D^{-1/2}AD^{-1/2}$. Let (γ_i, c_i) be the eigenpairs of N.

The Normal spectrum is $1= \gamma_0 = \max(\gamma_i) \geq \gamma_1 \geq ,..., \geq \gamma_{n-1} \geq -1$

Then calculate the pairs:

$$(\gamma_i\,, D^{-1/2}c_i) = (\nu_i\;, n_i)$$

We have that

$$n_i D n_j = \delta_{ij}$$

That is, the vectors are orthonormal in the D (or χ^2) metric. The Normal spectrum is called the Q-spectrum in CDS. The multiplicity of eigenvalue 1 is the number of components in G.

Compositions

The *Kronecker product* (tensor product) of two graphs G_1 and G_2 is most easily defined in terms of their adjacency matrices A_1 and A_2 as follows:

$$(A_1 \otimes A_2)_{ij} = A_2 \text{ when } A_1(i,j) = 1$$
$$= 0 \quad \text{otherwise, e.g.:}$$

$$\overset{A_1}{\begin{matrix}1&1&1\\1&0&0\\1&0&0\end{matrix}} \otimes \overset{A_2}{\begin{matrix}0&1&1\\1&0&1\\1&1&0\end{matrix}} = \overset{A_1\otimes A_2}{\begin{matrix}0&1&1&0&1&1&0&1&1\\1&0&1&1&0&1&1&0&1\\1&1&0&1&1&0&1&1&0\\0&1&1&0&0&0&0&0&0\\1&0&1&0&0&0&0&0&0\\1&1&0&0&0&0&0&0&0\\0&1&1&0&0&0&0&0&0\\1&0&1&0&0&0&0&0&0\\1&1&0&0&0&0&0&0&0\end{matrix}}$$

Note: this is a core-periphery structure, where $\mathbf{A}_1$ is a *blockmodel*.

The *Cartesian Product* may be defined in terms of the Kronecker product as:

$$A_1 \oplus A_2 = A_1 \otimes I_2 + A_2 \otimes I_1$$

where the I_i are identity matrices of size $|V_i|$.

Then we have that the Adjacency spectrum of $A_1 \oplus A_2$ is $\{\alpha_i + \beta_j\}$
where $\{\alpha_i\}$ is the spectrum of A_1 and $\{\beta_j\}$ is the spectrum of A_2
That is, the spectrum is the sum of all possible pairs.
Furthermore, the eigenvectors of $A_1 \oplus A_2$ belonging to $\{\alpha_i + \beta_j\}$ are Kronecker products of the corresponding eigenvectors of A_1 and A_2 , so that the eigenpairs are:

$$(\{\alpha_i + \beta_j\}\,, \{\, a_i \otimes b_j\,\})$$

We say the Adjacency spectrum *behaves well* under Cartesian product.
The Laplacian also behaves well under Cartesian product $L_1 \oplus L_2$ with eigenpairs:

$$(\{\lambda_i + \kappa_j\}\,, \{\, l_i \otimes k_j\,\})$$

Further, the eigenvalues of L_1 and L_2 always contain a $\lambda_0 = 0$ with corresponding constant eigenvector, so that the corresponding eigenpairs of $L_1 \oplus L_2$ are $(\lambda_1 + 0, l_1 \otimes 1)$. The term $l_1 \otimes 1$ means that the components of l_1 are *replicated* $|V_2|$ times. Since the Cartesian product of two paths is a grid, this produces a perfectly rectangular representation. The Laplacian is therefore a useful tool in problems involving regular grids (or hypergrids).

However, N does *not* behave well under Cartesian product.

The *Kronecker product* was defined above. It turns out that the Adjacency spectrum also behaves well under Kronecker product, so that $A_1 \otimes A_2$ has eigenpairs

$$(\{\alpha_i \times \beta_j\}, \{a_i \otimes b_j\})$$

The Laplacian does *not* behave well under Kronecker product.

However, the Normal spectrum *does* behave well under Kronecker product (Chow, 1997), so that $N_1 \otimes N_2$ has eigenpairs

$$(\{\nu_i \times \mu_j\}, \{n_i \otimes m_j\})$$

Further, the eigenvalues of N_1 and N_2 always contain a $\nu_0 = 1$ with corresponding constant eigenvector, so that the corresponding eigenpairs of $N_1 \otimes N_2$ are $(\nu_1 \times 1, n_1 \otimes 1)$. The term $n_1 \otimes 1$ means that the components of n_1 are *replicated* $|V_2|$ times. This produces *clustering* of the components of $N_1 \otimes N_2$ for these eigenvectors.

It appears that the behaviour under Kronecker product explains why both the Adjacency and Normal eigenvectors are good at detecting both on- and off-diagonal *blocks*. The signs of the Adjacency (after removing row-means) eigenvectors were studied by (Schwartz, 1977) and shown to be essentially those found by CONCOR.

Visualisation

The Laplacian can L provide good visual representations of graphs which are Cartesian products (such as grids and hypercubes); while N can provide good visual representations of graphs which are Kronecker products (such as graphs consisting of blocks). The reasons for this are suggested above and have mostly to do with the behaviour of eigenpairs which are sums and products with 0 and 1, respectively. For general (not k-regular) graphs, eigenpairs of A do not provide such good representations since, in general, there is no constant eigenvector to combine with.

Another way of describing these results is to consider the relationship between the eigenvector components for a node and those it is connected to. It is evident from the definition of eigendecomposition that (where u~v means u is connected to v)

$a_i(u) = \sum_{u \sim v} a_i(v)/\alpha_i$	for eigenpair i of A
$l_i(u) = \sum_{u \sim v} l_i(v)/(\lambda_i - \deg(u))$	for eigenpair i of L
$n_i(u) = \sum_{u \sim v} n_i(v)/(\nu_i \times \deg(u))$	for eigenpair i of N

Note that A has no control for node degree. Consider the effect for "important" eigenpairs ($|\alpha| \simeq k \gg 1$, $\lambda \simeq 0$ and $|\nu| \simeq 1$) when deg(u) is small: a(u) will be folded toward the origin, while l(u) and n(u) will sit further *away* from the origin than its neighbours. This effect makes it difficult to interpret visual representations based on A, except for k-regular graphs where all three spectra are essentially the same.

References

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