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REVIEW #5: ALGEBRA

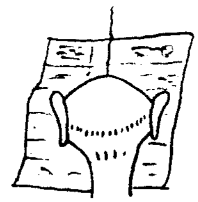
1 This is not a course in algebra. Most physics students have had that. I assume you have. But you still waste time (during physics exams!) doing simple things the hard way. The little investment in doing this review will pay you dividends in grades and time. NOW TURN THE PAGE. WE MOVE HORIZONTALLY.

8

$$d = \frac{b}{ab+1}$$

(By the way, do me a favor: Don't show this review to a mathematician. His hair would stand on end.)

9



Try some moves with the following,

$$\frac{\sqrt{2}\,a+5}{7b^2} = \frac{mc^2}{14b}$$

DANGER !

so as to get c^2 alone on one side.

16

$$25.4\ \cancel{km} \times \frac{100\,000\ \cancel{cm}}{1\ \cancel{km}} \times \frac{1\ in}{2.54\ \cancel{cm}}$$

$$= 1{,}000{,}000\ in$$

17

Now see if you can pass the test.
Given that 292 pennyweights (dwt) = 256 drams,
and that 16 drams = 1 oz,
and that 1 oz = 480 grains,
convert 1 dwt into grains.

1 dwt x____ x____ x____ = $\boxed{}$ grains.

24

You can do your "processing" much better after all the statements of fact are neatly lined up before you, in the form of equations. School teachers invent queer gimicks to

25

cut down the number of equations. We want to do the opposite. Stamp out sentences and get more equations! Write, below, all the equations you can get out of this worn out old story: "Three boys together earned \$60. Henry earned \$2 less than Ed. Jack earned twice as much as Henry."

32

How are you holding up? Can you stand a couple more ideas before we start recapitulating?

33

DIVISION OF AN EQUATION BY AN EQUATION may not be a proper name for it, but it is a most useful maneuver.

If $ab = cd$
and $ef = gh$
then $\frac{ab}{ef} = \frac{cd}{gh}$

Now you do the same with these:

If $4x = 3yz$
and $2x = 6y$
Then —— = ——

40

The first one.

$$m = \frac{m}{\cancel{sec^2}}\ \cancel{sec^2}$$

cancellation leaves meters = meters.

41

What if you remember that $\sqrt{2as}$ is something but don't remember just what. Will putting the units in give you a clue?

$$\sqrt{\frac{meters}{sec^2}\ meters} = ?$$

POSTSCRIPT: This is something extra, but none the less valuable, in my opinion. It has to do with the tricks you can play with numbers that are very little different from one. A number close to one would be written 1+b or 1-b, where b is small compared to 1. b is just that little difference, one way or the other.

We are interested in the fact that

$$\frac{1}{1+b} \stackrel{a}{=} 1-b \quad and \quad \frac{1}{1-b} \stackrel{a}{=} 1+b$$

where the little a signifies "approximately equal to". But how approximate is approximate? I would like to have you set

go →

1

The tricks I will show you aren't new. I'll be "teaching new dogs old tricks." (I've given up the converse.)

2

I will be asking you to do some little operation in nearly every frame. Don't just think the answer; do what is asked <u>in pencil</u>, right in the space provided. Move on now, horizontally.

9

$$c^2 = \frac{2(\sqrt{2}a+5)}{mb}$$

Did <u>you</u> get c^2 on the <u>right</u>? No strain. Just <u>switch</u> everything side for side. (Equal signs work both ways, you know!)

10

When you were a small potato in school you used to work a gambit called "cross multiplying". You changed

$$\frac{A}{B} = \frac{C}{D} \text{ to } AD = BC$$

17

$$1\,\cancel{dat} \times \frac{256\,\cancel{dr}}{292\,\cancel{dat}} \times \frac{1\,\cancel{oz}}{16\,\cancel{dr}} \times \frac{480\,gr}{1\,\cancel{oz}}$$

$$= 26.2\ gr$$

18

The same scheme works when units are squared, cubed, etc. Suppose we want to convert a volume of 12 cubic ft (written 12 ft^3) to cubic cm (written cm^3), knowing that 1 ft = 30 cm.

You multiply by $\frac{30\text{ cm}}{1\text{ ft}}$ <u>as many times as you need to</u>

$$12\ ft^3 \times \qquad = \boxed{}\ cm^3$$

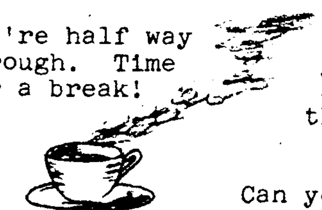
25

$$H + E + J = 60$$

$$H = E - 2$$

$$J = 2H$$

26



We're half way through. Time for a break!

Next we are going to look at how you solve the groups of equations after you have fished them out of the sentences. A while ago we had

$$J = M + 3 \text{ and } M = \frac{5}{8}J$$

Can you "get rid of" M by a substitution? Do it.

33

$$\frac{4x}{2x} = \frac{3yz}{6y}$$

or, $2 = \frac{z}{2}$

34

Let's look at $W=\frac{1}{2}kX^2$. (This is the formula for the work, W, done in stretching a spring a distance X.) Here is a story problem: "If W is 5 when X is 3, what is W when X is 5?" Take the first half first: "If W is 5 when X is 3." Substitute these values into $W=\frac{1}{2}kX^2$ and you get your first equation. It is:

41

$\frac{\text{meters}}{\text{sec}}$,

which is velocity.

Now it comes back!

$$v = \sqrt{2as}$$

42

If you remember that in every equation you write (if correct) the <u>units</u> on the two sides are the same, you will use that fact to help yourself out of a hole every now and then.

Let's recapitulate briefly.

b = .05, so that you have

$\frac{1}{1.05}$ and $\frac{1}{.95}$

and then I would like to have you do the long divisions:

$$1.05\overline{)1.00000}$$

and $.95\overline{)1.00000}$

Here is space for it.

2 Each time put your score in the small square.

0 if you miss
2 if correct
1 if in between

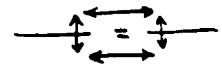
Now proceed

HERE

3 Look at this familiar type: $\frac{abc}{def} = \frac{ghi}{klm}$

You can move a symbol in these ways:

You can cancel in these ways:



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10 Cross multiplying is handy when you have something like:

$$\frac{\frac{ab}{cj}}{\frac{g}{e}} = \frac{\frac{d}{fh}}{\frac{i}{k}}$$

11 First you can cross multiply it to get: $\frac{abi}{cjk} = \frac{dg}{fhe}$.

Try this: $\frac{\frac{2l}{9}}{\frac{1}{2}} = \frac{\frac{1}{3}}{\frac{x}{y}}$ Do it here:

18

$$12\,\cancel{ft^3} \times \frac{30\,cm}{1\,\cancel{ft}} \times \frac{30\,cm}{1\,\cancel{ft}} \times \frac{30\,cm}{1\,\cancel{ft}}$$

$$= 324{,}000\ cm^3$$

19 I started by mentioning drachms to dekasteres. It should be easy now. How many dekasteres is 10 drachms? (1 dekastere = 10 cu meters; 1 oz = 30 cu cm; 1 drachm = 1/8 oz; 100 cm = 1 meter.) Get this and you get an apothecary's license.

26

$$J = \frac{5}{8}J + 3$$

which boils down to

$$J = 8$$

27 Here are the two we started with, just for ready reference:

$$J = M + 3$$

$$M = \frac{5}{8}J$$

Now that you know that J=8, find what M is. Surprisingly, you can do this from <u>either</u> of the original equations

$$M = ?$$

34

$$5 = \frac{1}{2}k\,3^2$$

OK, now for the second half: "What is W when X is 5?" (You don't have a value for W,

35 so it has to stay W for the present; in fact W is the unknown.) Anyway, substituting into $W=\frac{1}{2}kX^2$ you get your second equation, which is:

$$W = \ ?$$

Now dividing your second equation by your first:

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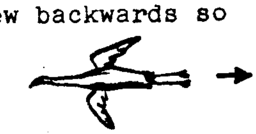
42 Funny word, recapitulate. Doesn't mean "surrender again." For the rest of the frames we will be like the bird who always

43 flew backwards so



he could see where he had been.

Get a on the left side, alone.

$$\frac{2bcX^2\sqrt{2}}{3c^2} = \frac{b^2a}{\sqrt{2}\,X^2}$$

You got .9524 and 1.0526. Pretty close to .95 and 1.05. Therefore the "approximation" would have been plenty good in a problem requiring only 2 place accuracy.

What about b = .01? You would have gotten, by long division, .99009+ and 1.0101+.

Extremely close to 1-.01 and 1+.01.

More than good enough for 3 place accuracy.

Going to a larger b, if b=.1, the result is not very good:
.9 and 1.1
by the approximation;
.909+ and 1.11+
by long division.
So we shouldn't work with b as big as .1.

3

MOVE

CANCEL

Like learning the rules of checkers

4

Now you start doing the work. In the equation below, move things around and cancel so as to end up with only d on the left side.

$$\frac{gb}{ecaf} = \frac{bcae}{dfg}$$

11

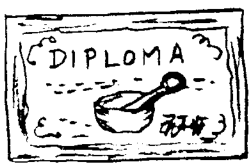
$$\frac{2lx}{9y} = \frac{1}{6}$$

12

One final example, for you to cross multiply. It's a little different.

$$\frac{\frac{4\sqrt{n}}{2}}{\frac{\pi}{3}} = \frac{\pi^2}{\sqrt{n}}$$

19 *Ans:* $\frac{3}{800{,}000}$ *dekasteres*



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STORY PROBLEMS were invented to make school children suffer. But you will never see the last of them. Most of your physics problems will be "story problems." The hard part is translating from the word language into mathematical language. In other words to get the problem out of the story form as fast as possible! That's 3/4 of the battle.

27

M = 5

28

You had two separate pieces of information about J and M (in the form of two equations) and you had to "use up" both of them, in the process of finding the values of the two unknowns J and M. Do you remember the general rule about the number of equations needed for evaluating a given number of unknowns?

35

$$\frac{W}{5} = \frac{\cancel{\tfrac{1}{2}k}\,5^2}{\cancel{\tfrac{1}{2}k}\,3^2}$$

or, $W = \frac{125}{9}$

36

Now we'll handle both halves of a "story" in one frame. "If 3 ft. lb work is done in stretching a certain spring 2 ft, how much work is done in stretching it 2.5 ft? (Still using $W = \frac{1}{2}kX^2$.)

#1:

#2:

Solution by dividing:

43

$$a = \frac{4x^4}{3bc}$$

44

If the acceleration of gravity is g = 9.8 meters/sec^2, what is it in miles/hr^2 ? Use: 1 mi = 5280 ft and 1 m = 3.2 ft. You know how many sec = 1 hr. (we hope!)

True, you say, but how often do we meet a number that is that close to 1? You'll find quite a lot, if you look for them.

Would you evaluate $\frac{400}{398}$ by long division?

No, you recognize it as a number close to 1, and write it as:

$$\frac{400}{400-2} = \frac{\cancel{400}}{\cancel{400}\left(1-\frac{2}{400}\right)} \doteq 1 + \frac{2}{400}$$

Time for you to do a few. Change each of these into a decimal number, using our approximation method.

$\frac{1}{1.0032} \doteq$

$\frac{1}{.97} \doteq$

$\frac{250}{245} \doteq$

4

$$d = \frac{e^2c^2a^2}{g^2}$$

5

When you move something, cross it out of the old place boldly and write it in its new place. When you cancel, do that boldly too. No pussyfooting!

Do this one, leaving w and nothing else on the left side.

$$\frac{3wz}{10} = \frac{3y}{5}$$

12

$$\frac{4\sqrt{n}\sqrt{n}}{2} = \frac{\pi^3}{3}$$

or, $2n = \frac{\pi^3}{3}$

13

Enough moving and cancelling for now. How are you at converting from drachms to dekasteres? Try not to be bored while we start real simple and convert 15 ft into inches. 12 in = 1 ft, so we can write that:

$$\frac{12\text{ in}}{1\text{ ft}} = 1 \quad \text{and} \quad \frac{1\text{ ft}}{12\text{ in}} = 1$$

No ques. go →

20

To dissect a "story" you need a sharp eye. Let me give your eye a quick test. What's wrong with the triangular sign at the right?

21

Maybe you'll do better with this: How many times 4 is 12? Translate it into an equation. "How many" will have to be a symbol--say H. "Is" usually translates into =. "Times" is x. What do you get?

28

You need as many equations as there are unknowns.

29

But that's not quite all there is to it. Let me expose you to a baited trap. Can this group of equations be solved for all the unknowns? (I don't mean do it; just say whether it looks possible.)

$$\left\{\begin{array}{l} x + 2y + z = 0 \\ x + y \quad\;\; = 5 \end{array}\right\} \quad \left\{\begin{array}{l} w + 2y \quad\;\; = 8 \\ w \quad\;\; + 3z = 4 \end{array}\right\}$$

36

$$\frac{W}{3\text{ ft lb}} = \frac{\cancel{\tfrac{1}{2}}\cancel{k}\, 2.5^2}{\cancel{\tfrac{1}{2}}\cancel{k}\, 2^2}$$

or, $W = \frac{3 \times 6.25}{4}$ ft lb

37

The period, T, of a pendulum is $T = 2\pi\sqrt{\frac{\ell}{g}}$. If a certain pendulum has a period 2 sec, what would its period be if its length (ℓ) were multiplied by 2.5 and g were multiplied by 8. Insert these values at right:

#1: $2\text{ sec} = 2\pi\sqrt{\frac{\ell}{g}}$

#2: $T' = 2\pi\sqrt{\quad ?}$

44

$$\frac{9.8\,\cancel{m}}{\cancel{sec^2}} \times \frac{3.2\,\cancel{ft}}{1\,\cancel{m}} \times \frac{1\text{ mi}}{5280\,\cancel{ft}} \times \frac{3600\,\cancel{sec}}{1\text{ hr}} \times \frac{3600\,\cancel{sec}}{1\text{ hr}}$$

$$= \frac{9.8 \times 3.2 \times 3600^2}{5280}\ \text{mi}/\text{hr}^2$$

45

Here is a "story." Put it into equation form. A man decides to shoot his wad of $100 on shirts and hats, and finds he can get 5 hats and 10 shirts or 6 hats and 8 shirts for exactly $100. We write:

$100 = ? and $100 = ?

Answers:

.9968

1.03

1.02

Beats long division!

Powers and roots of numbers that are close to 1 are just as easy as what we have been doing. We can write a pair of formulas to cover all situations:

#1 $(1+b)^k \doteq 1 + kb$

#2 $(1-b)^k \doteq 1 - kb$

These have abundant possibilities. Study these examples carefully: →

$(1-b)^2 \doteq 1 - 2b$ (#2, k = 2)

$\frac{1}{(1-b)^2} \doteq 1 + 2b$ (#2, k = -2)

$\sqrt{1+b} \doteq 1 + \frac{1}{2}b$ (#1, k = $\frac{1}{2}$)

$\frac{1}{\sqrt{1-b}} \doteq 1 + \frac{1}{2}b$ (#2, k = $-\frac{1}{2}$)

$\frac{1}{1+b} \doteq 1 - b$ (#1, k = -1)

5

Like this:

$$\frac{\cancel{3}w\cancel{z}}{\cancel{10}} = \frac{\cancel{3}y}{z\cancel{5}}\,\overset{2}{\cancel{10}}$$

$$w = \frac{2y}{z}$$

6

Here's a more complicated one, for practice. Get b alone on the right side this time.

$$\frac{2e^2 b\sqrt{n}}{b^2 a} = \frac{28\pi a^3}{7}$$

13

You also know that you are privileged to multiply anything by 1, at any time.

14 So in attempting to convert 15 ft into inches, we can multiply it either by $\frac{12\text{ in}}{1\text{ ft}}$ or by $\frac{1\text{ ft}}{12\text{ in}}$. (Both = 1.) But which of them will get us where we want to go?

$15\text{ ft} \times \frac{1\text{ ft}}{12\text{ in}}$? NO! $15\text{ ft} \times \frac{12\text{ in}}{1\text{ ft}}$? YES! How did we know?

21

$$H \times 4 = 12$$

22

To nail it down, do this one: "Joe's age is 3 years greater than Mary's age." You will need symbols for Joe's and Mary's ages; call them J and M. Do it over there →

29

Yes. 4 equations, 4 unknowns.

30

Are there enough equations here for a solution? Look them over carefully!

$$X + 2y + Z = 0$$
$$2X + 3y + Z = 0$$
$$4X + 6y + 2Z = 0$$

37

$$\#2:\ T' = 2\pi\sqrt{\frac{2.5\ell}{8g}}$$

$$\#1:\ 2\text{ sec} = 2\pi\sqrt{\frac{\ell}{g}}$$

38

Since we would like to have T come out to be in the numerator on the left side, let's divide, by putting #2 over #1. After you have done this, cancel as much as you can.

45

$$\$100 = 5H + 10S$$
$$\$100 = 6H + 8S$$

So, $S = \$5$

$H = \$10$

46 The period of a pendulum is $T = 2\pi\sqrt{\frac{\ell}{g}}$. On Venus a certain pendulum has a period of 2 sec. What will its period be on the moon, where g is 0.13 times as much as it is on Venus? Write the two equations; divide one by the other, and cancel like mad.

So you see how it goes. 10 minutes ago could you have told me how much $\frac{1}{\sqrt{.97}}$ is? I did it in my head and got 1.015. I thought of it as $(1-.03)^{-\frac{1}{2}}$; used #2 with $k = -\frac{1}{2}$. This gave $1 + \frac{1}{2}\cdot .03 = 1.015$.

Let's you try these:

$(.98)^3 \doteq$

$\frac{1}{(1.02)^2} \doteq$

$\sqrt{1.01} \doteq$

6

$$\frac{e^2\sqrt{n}}{2\pi a^4} = b$$

7

Now I'm going to wave a big red flag! If there are + or - signs lurking around, watch out! Example: $\frac{ab+1}{ab} = \frac{b}{abd}$

You can't move or cancel the ab independently of the 1, or vice versa.

14

Because we can cancel the <u>units</u>.

$$15\,\cancel{ft} \times \frac{12\text{ in}}{1\,\cancel{ft}} = 180\text{ in}$$

15

Convert 13 meters into inches, using the fact that 1 m = 39 in. Use the method we have just illustrated, even if you <u>can</u> do it in your head. I know it's simple but I <u>want</u> you to do it my way.

22

J = 3 + M

23

The plot thickens when the sentence contains enough information to make two or more equations. Find the two separate equations in this: Joe's age is 3 yrs greater than Mary's age and Mary is 5/8 as old as Joe is.

Do it here.

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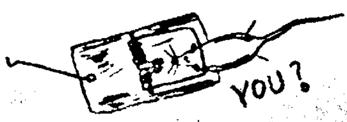
No. The last one was just a multiple of the second one.



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There was <u>redundancy</u>; that is, two of them said the same thing. Another way to say it is that we did not have 3 independent equations. Now, can you state a more water-tight rule concerning the number of equations and number of unknowns?

38

$$\frac{T'}{2\text{ sec}} = \frac{\cancel{2\pi}\sqrt{\frac{2.5\cancel{g}}{8\cancel{g}}}}{\cancel{2\pi}\sqrt{\frac{\cancel{g}}{\cancel{g}}}}$$

or, $T' = 2\sqrt{\frac{2.5}{8}}\text{ sec}$

39

The period of a satellite can be found from $T^2GM = 4\pi^2R^3$ where R is the radius of its orbit, M is the mass of the earth, and G is the gravitational constant. Find the ratio of the periods of two satellites; one in orbit at 10,000 mi radius; the other in orbit at 15,000 mi radius.

46

$$\frac{T_M}{2\text{ sec}} = \frac{\cancel{2\pi}\sqrt{\frac{\cancel{g}}{0.13\cancel{g}}}}{\cancel{2\pi}\sqrt{\frac{\cancel{g}}{\cancel{g}}}} = \sqrt{\frac{1}{0.13}}$$

so, $T_M = \frac{2}{\sqrt{0.13}}\text{ sec.}$

47

Given that:
g is the acceleration of gravity in meters/sec^2
v is velocity in meters/sec
h is height in meters.
I have forgotten whether $gh = \frac{1}{2}v$ or $gh = \frac{1}{2}v^2$.
Can you show me which one has to be right?

Answers:

$$(.98)^3 = (1-.02)^3 \doteq .94$$

$$\frac{1}{(1.02)^2} = (1+.02)^{-2} \doteq .96$$

$$\sqrt{1.01} = (1+.01)^{\frac{1}{2}} \doteq 1.005$$

How accurate are these approximations? For comparison, I did the first and second by multiplication and long division. I did the third by finding a 6-place table of square roots. The whole business took me 20 minutes.

I got:

.941192

.961168+

1.004988+

Pretty close!

7

The individual symbols (a, b, etc.) can be moved or cancelled these ways:

$$\frac{ab+1}{ab} \qquad \frac{b}{abd}$$

MOVE OK

CANCEL OK

CANCEL OK

8

But you can move or cancel the ab+1 as a whole if you keep it intact as (ab+1). Now, looking at the same thing again,

$$\frac{ab+1}{ab} = \frac{b}{abd}$$

Move and cancel to get d alone on the left side.

15

$$13\,\cancel{m} \times \frac{39\ in}{1\,\cancel{m}} = 507\ in$$

16

For this one you will multiply by two things that are equal to 1. Convert 25.4 km to inches, using the following facts: 1 in = 2.54 cm, and 1 Km = 100,000 cm. (Don't shortcut!)

23

J = 3 + M

M = 5/8 J

24

This is a message I don't want you to miss: When a "story" contains the makings of several separate algebraic statements (equations), extract all of them and write them down, no matter how trivial some of them may seem. Don't try to do some of the substituting, simplifying, or digesting in your head before you write the equation. That may economize on pencils, but it will invite mistakes.

(No ques; move along)

31

The number of independent equations must be as great as the number of unknowns.

32

Let me tell you something about writers of story problems. They don't have much imagination. You can count on them to make the number of unknowns just equal to the number of possible equations. Therefore you must always be sure to squeeze all the equations out of the sentences. If you miss one you won't get a solution.

39

$$\frac{T_1^2 \cancel{GM}}{T_2^2 \cancel{GM}} = \frac{\cancel{4\pi^2}\ 10,\cancel{000}^3}{\cancel{4\pi^2}\ 15,\cancel{000}^3}$$

$$\frac{T_1}{T_2} = \sqrt{\frac{10^3}{15^3}} = \frac{1}{\sqrt{1.5^3}}$$

40

Any true equation must have the same units on both sides. Suppose you forgot which of these is correct: $s = \frac{1}{2}at^2$ or $s = \frac{1}{2}at$ (s is distance in meters; a is acceleration in meters/sec^2 ; t is time in sec.) Substituting just the units can you check which is correct?

First one:

$$m = \frac{m}{sec^2}\, sec^2$$

Second one:

$$m = \frac{m}{sec^2}\, sec$$

Which is true?

47

$$\frac{meters}{sec^2} \times meters = \left(\frac{meters}{sec}\right)^2 \text{ OK!}$$

Once a dentist was about to administer gas to a patient to ease the agony of extract-

48

ing a tooth. The patient asked, "Will I know anything The dentist replied, "That would be asking too much of the gas. It is only guaranteed to relieve the pain a little." Don't expect much more than that of this brief review. If it relieves a little pain on some future quiz, I will feel gratified. THANKS AND GOOD LUCK! That's all except for the POSTSCRIPT--if you have stamina left. (A perfect score is 74. How did you do?)

If you like things neatly wrapped up, then everything we have mentioned in the entire postscript is covered by

$$(1 \pm b)^k \mathrel{\hat{=}} 1 \pm kb.$$

Check it and see. Now in the little space left I will show you how higher mathematics--the branch called topology--can be employed to out-wit a chicken snake. Just in case you ever have such a problem!