Vectors

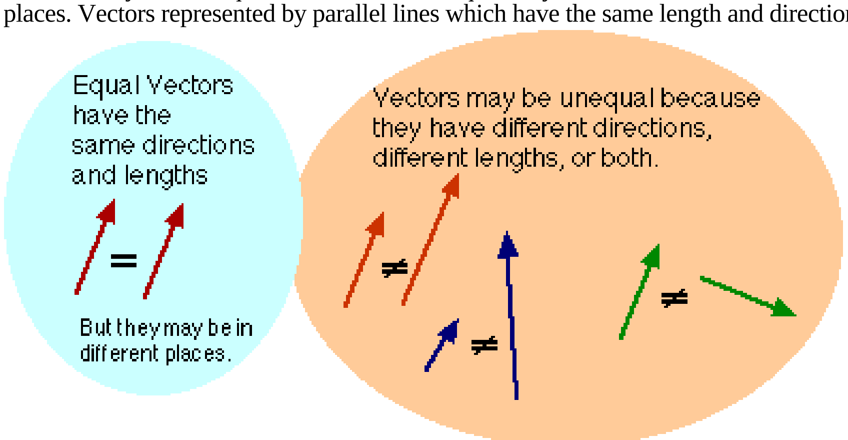
If our aim is to describe and understand motion, then just driving up and down along a line is not going be enough. Motion in the real world happens in three dimensions. We need a method of describing motion a complicated as that of a bee or a helicopter toy. The way we do that is with vectors.

In the beginning I want you to think about vectors as geometrical objects. The model for all vectors is displacement. Something starts in one place and ends up in another. One can stretch an elastic cord between two points in space and mark which end was the destination by a little paper arrow. You can see that there are two aspects to this displacement which we call a vector. One is that it has a length, or magnitude. Secondly it has a direction: something to do with its angle. The first item on our agenda of describing vectors is to figure out when two vectors are equal.

Let''s make another little trip the same length as the first but taking off in another direction. If you stretch a cord between the starting and ending locations you can visualize two vectors. Are they equal? Take another trip going in the same direction as the first one, but now going farther. This is another vector. Is it equal to the first?

When are vectors equal?

In both cases, when vectors have either different directions, or different lengths, they are not equal. Likewise, if both direction and length are different they are not equal. Two vectors are equal only when both directions and lengths are the same. Two vectors can be equal, but in different places. Vectors represented by parallel lines which have the same length and direction are equal.

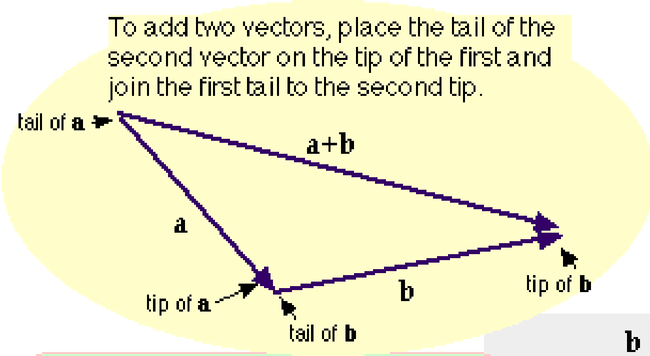


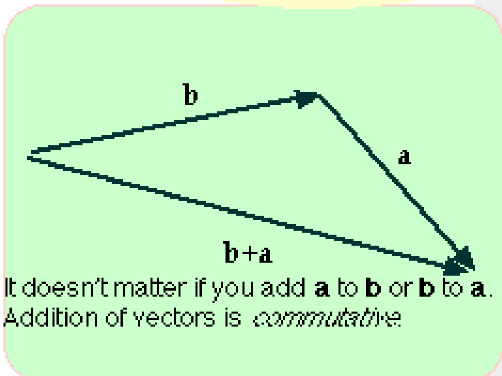
How you add vectors

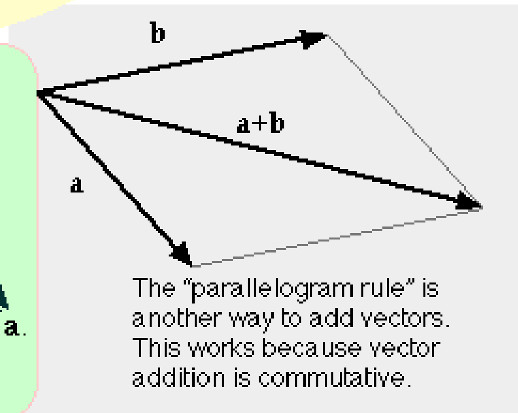
When something makes one displacement, then another, the total displacement is the sum of the two individual ones. Let **a** represent the first trip, and **b** the second trip. The total trip is given by lining up the vectors representing the trips tip-to-tail. Put the tail of the second on top of the tail of the first. The total displacement is taken to be the vector from the tail of the first one to the tip of the second one. This way of adding vectors comes naturally when we are talking of displacements. Sometimes other vector quantities, such as forces, must be added. Then it is often necessary to move one vector so that its tail lines up with the other's tail because force diagrams usually show forces with all their tails together on the object to which the force is applied. When a vector is moved remember to keep its length and direction the same.

Commutativity of Addition

The sum of vectors can be written as **c** = **a** + **b**. This means that the tail of b is placed on the tip of **a** and the tail of **c** is on the tail of **a** and the tip of **c** is on the tip of **b**. It is also possible to add **a** to **b**. Is the result the same as adding **b** to **a**? If displacement is our model vector, then it would be the same result no matter which trip is taken first. So we will require vectors to give the same sum no matter which is added first. **a**+**b** = **b**+**a**. This property is called commutativity.

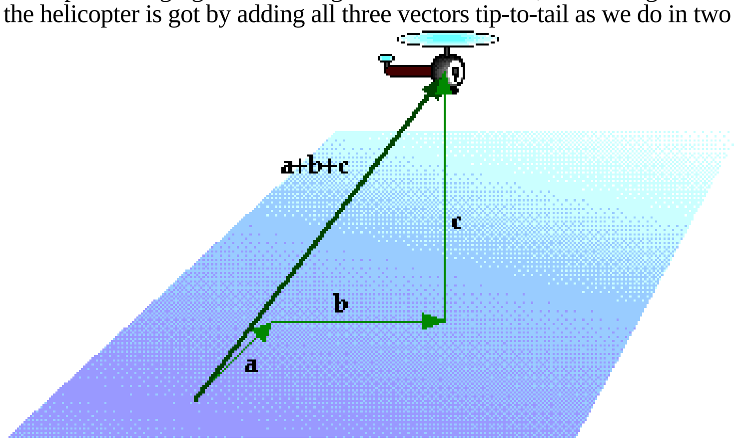




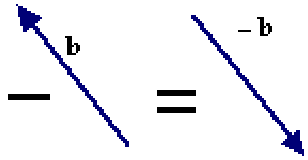


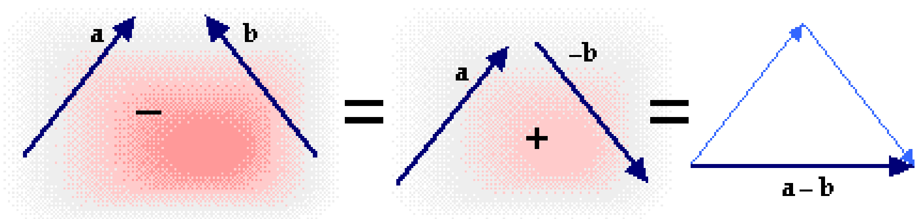
Not all quantities which can be assigned a size and direction have this property, and thus those noncommutative quantities are not allowed to be vectors. (It's a club with high standards!) For example rotations about an axis have a size, the angle of rotation, and a direction, the direction of the axis of rotation. Try rotating a book first 90° about a vertical axis and then 90° about the horizontal axis going left-right. Reverse the order of these two rotations and you end up with the book in a different position. In the case of rotations, **a**+**b** is not equal to **b**+**a**.

It's easy to draw vectors on paper to help you visualize the process of vector addition. One should also try to imagine vectors in three dimensions. Everything we say about two-dimensional vectors, can be extended to three dimensions without too much difficulty. Imagine a helicopter emerging from its hanger. It taxis 100 m N, then changes direction, taxis 200 m E and then goes up 300 m. The total displacement of the helicopter is got by adding all three vectors tip-to-tail as we do in two dimensions.

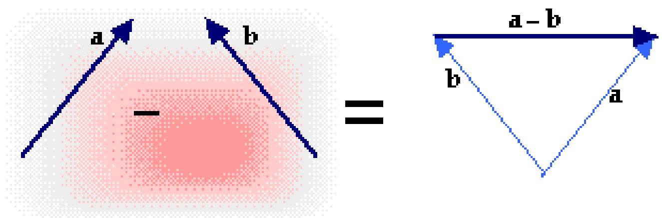


The negative of a vector and subtraction





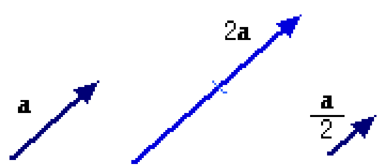
Another way to find **a** – **b** is to place the tails of **a** and **b** together and draw a vector from the tip of **b** to the tip of **a**. This method means finding the vector which you add to **b** to get **a.** (That's **a** = **b** + (**a** – **b**))



Try both methods of subtraction and you will see that they give a vector with the same magnitude and direction. (Though they will be in different places after the construction, but that doesn't matter.) You can choose whichever method of subtraction which seems to make more sense to you.

Multiplication by a scalar

Sometimes vectors are multiplied or divided by a scale factor. For example, if one stretches an elastic cord representing a vector to twice its length, keeping its direction the same, then the original vector is multiplied by two. Similarly, if one lets it reduce to half its length it is divided by two, or multiplied by one-half. whichever you like. Ordinary numbers are called scalars to distinguish them from vectors. You've been using scalars since you learned to count.



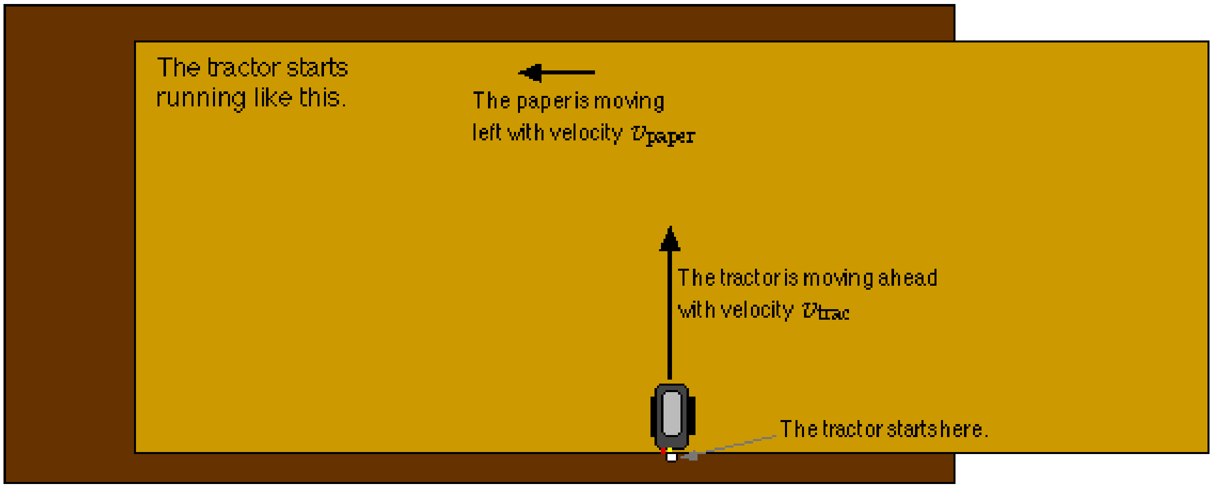
There are also operations involving two vectors called vector multiplication. These we will try to avoid in this course but you may need to learn them if you take further physics or mathematics courses.

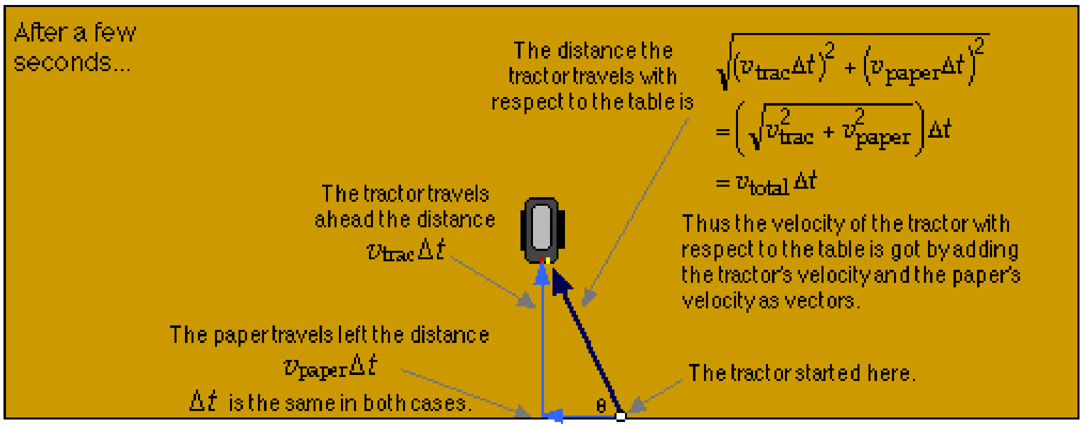
Quantities like displacement, velocity, acceleration and force are vectors. Mass and time are scalars, as are quantities like temperature, humidity and energy.

Velocity and acceleration vectors

Displacement is our prototype vector. Velocity is derived from displacement by dividing it by an interval of time. The result of dividing a vector by a scalar is a vector. Likewise, dividing a vector representing velocity difference by a time interval gives a vector--the acceleration. Mass, a scalar, multiplying acceleration, a vector, gives force, another vector quantity.

To see how velocities can add like vectors I use a toy tractor. It sits on a long piece of butcher paper covering the table. Let's say that the table length is oriented east-west. The table is about 1 m wide. If I set it going across the table northwards, it covers the 1 m distance in 10 seconds. That's a velocity of 0.1 m/s, north. The second time I let the tractor cross the table, I pull the butcher block paper at about 0.05 m/s, east, half the speed of the tractor and at right angles to its heading. This time the tractor crosses the paper directly, but, relative to the table, it crosses at an angle. The velocity of the tractor with respect to the table is no longer 0.1 m/s, but the vector sum of 0.1 m/s, north and 0.5 m/s, east. That adds





The angle at which the tractor head with respect to the table is given by trigonometry

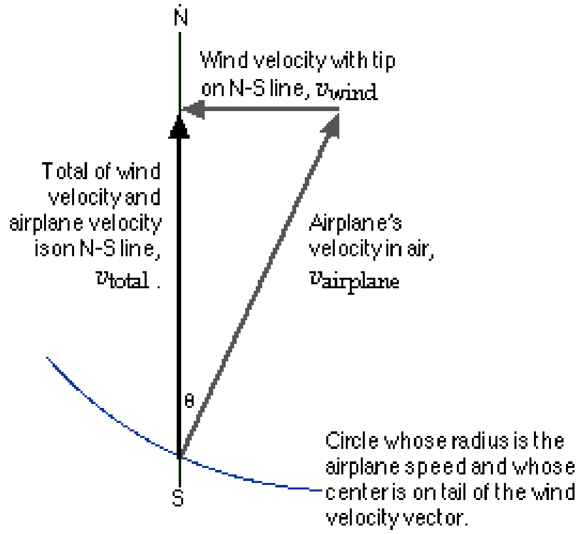
$$\tan\theta = \frac{v_{trac}}{v_{paper}}$$

$$\theta = \tan^{-1}\left(\frac{v_{trac}}{v_{paper}}\right)$$

Navigation

The next problem would be to find out how to achieve a desired direction when a cross wind is blowing. For example, say we want to travel straight north in an airplane, but there is a cross wind blowing from west to east. What direction should the plane head so that it ends up travelling due north? Let's say the cross wind blows at 40 km/h and that the plane travels at 100 km/h with respect to the air. Do the problem graphically as follows:

1. Draw a straight line north-south, along the desired heading.
2. Draw a vector to scale representing the wind velocity: 40 km/h eastwards with it's tip on the north-south line.
3. The airplane's velocity is 100 km/h, but we don't know what direction. Place a vector with it's tip on the tail of the wind speed vector and find the angle so that the tail of the airplane velocity's vector is on the north-south line. Now you have a vector, 100 km/h long and pointing in the direction so that when it is added to the wind velocity gives a total velocity due north.
4. To find the airplane velocity with respect to the ground, measure from the tail of the airplane's velocity vector to the tip of the wind velocity's vector. Or you can use Pythagoras' theorem.



$$v_{total} = \sqrt{v_{airplane}^2 - v_{wind}^2}$$

$$\theta = \sin^{-1}\left(\frac{v_{wind}}{v_{airplane}}\right)$$

In this case the heading which allows the airplane to go due north is 23.6° east of north and the speed of the airplane with respect to the ground is 92 km/h.

Acceleration

Acceleration is also a vector because it is the difference of two velocities (a vector) divided by a time interval (a scalar). The result is a vector. Newton's law says that force is mass times acceleration. Mass is a scalar, acceleration is a vector; therefore, force is a vector too.

What is surprising about acceleration and force vectors is that they do not necessarily point in the direction of the motion. We'll find that out when we talk about curvilinear motion, and in particular, motion in a circle.