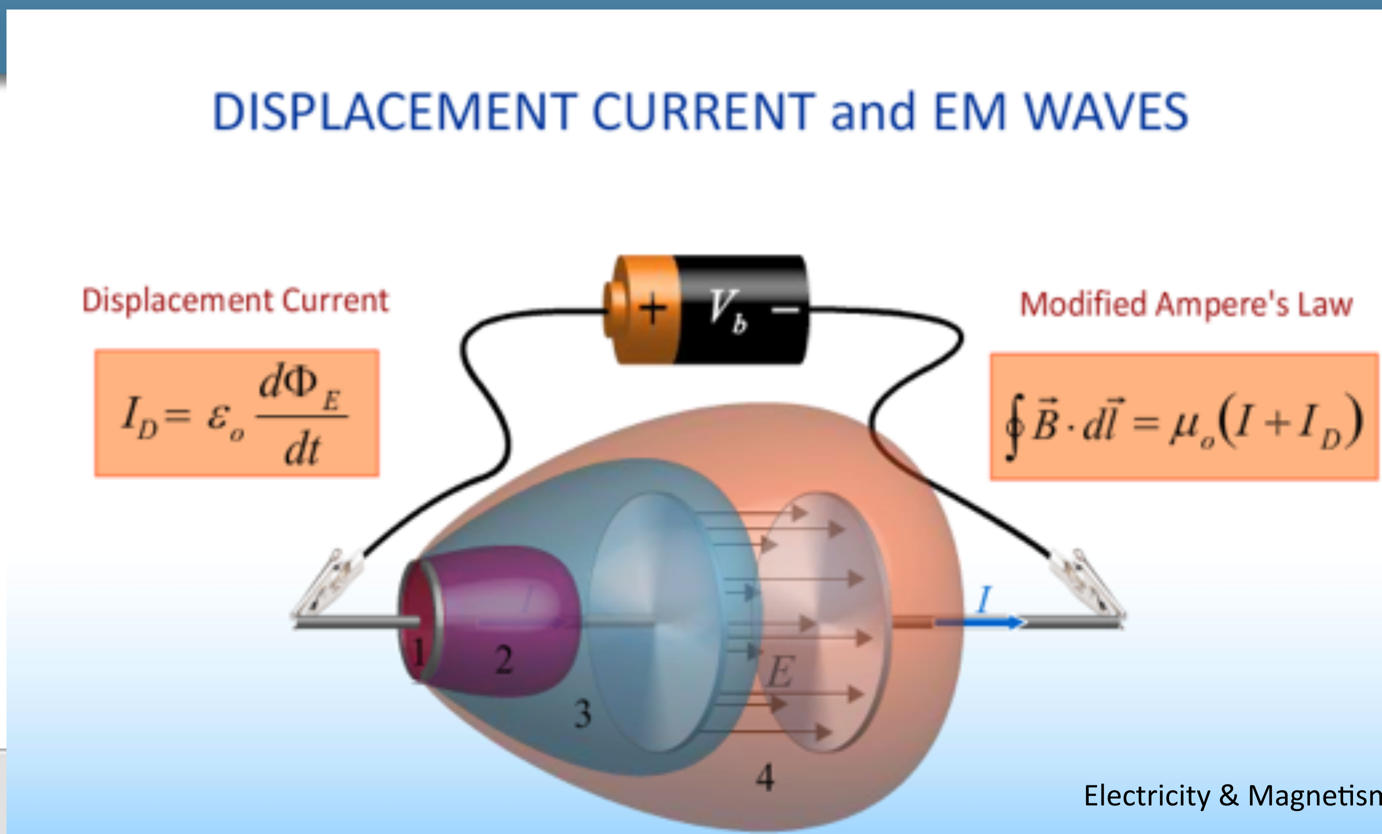
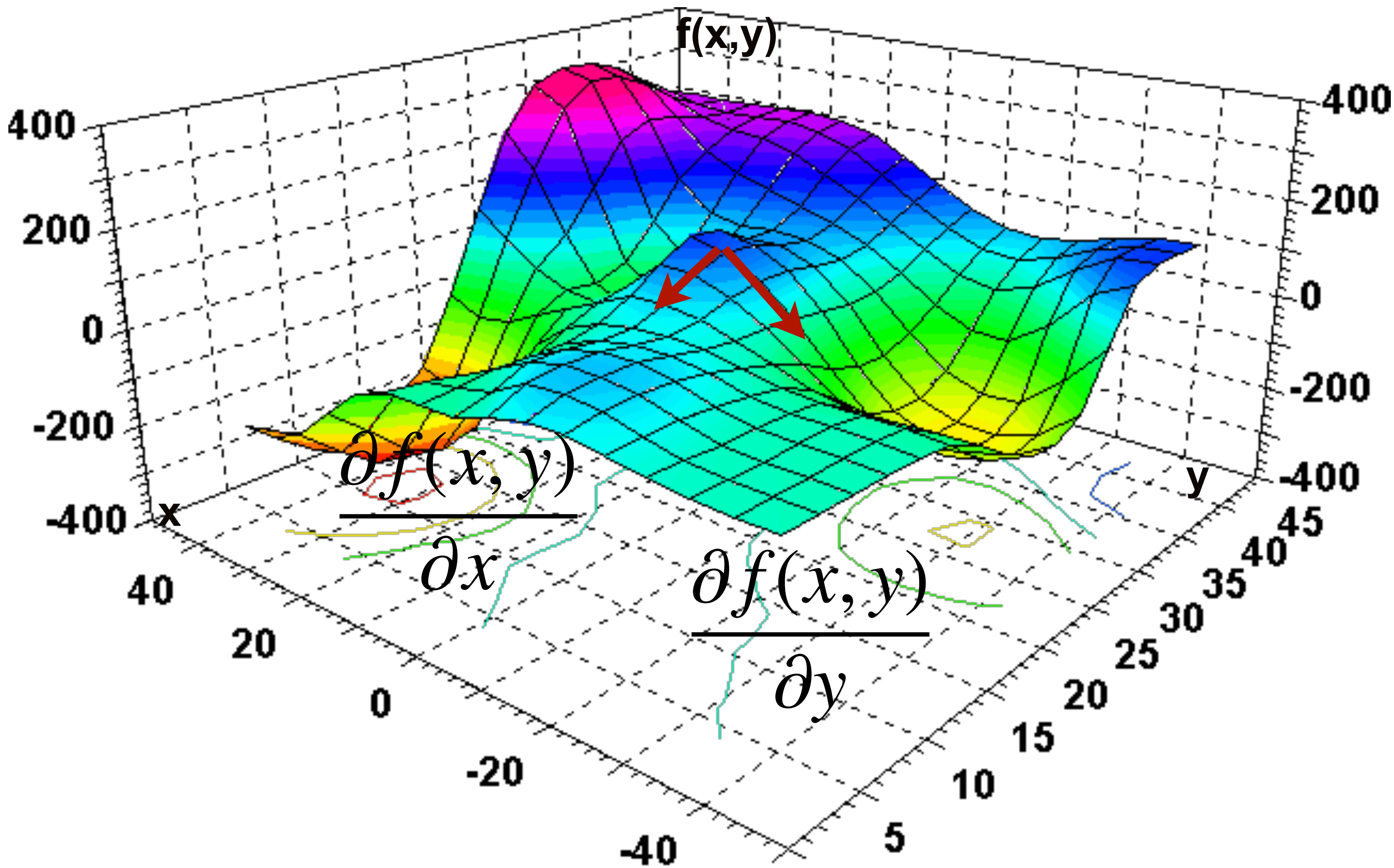
Electricity & Magnetism
Lecture 22




f(x,y)
400
200
0
-200
-400
x
40
20
0
-20
-40
y
5
10
15
20
25
30
35
40
45
$\frac{\partial f(x,y)}{\partial x}$
$\frac{\partial f(x,y)}{\partial y}$

Mechanical Universe and Beyond

Episode on Maxwell's Equations

- https://youtu.be/SS4tcajTsW8
- Historical context
- Visual animations

What We Knew Before Prelecture 22

Maxwell's Equations

Gauss' Law for E Fields

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_o}$$

Gauss' Law for B Fields

$$\oint \vec{B} \cdot d\vec{A} = 0$$

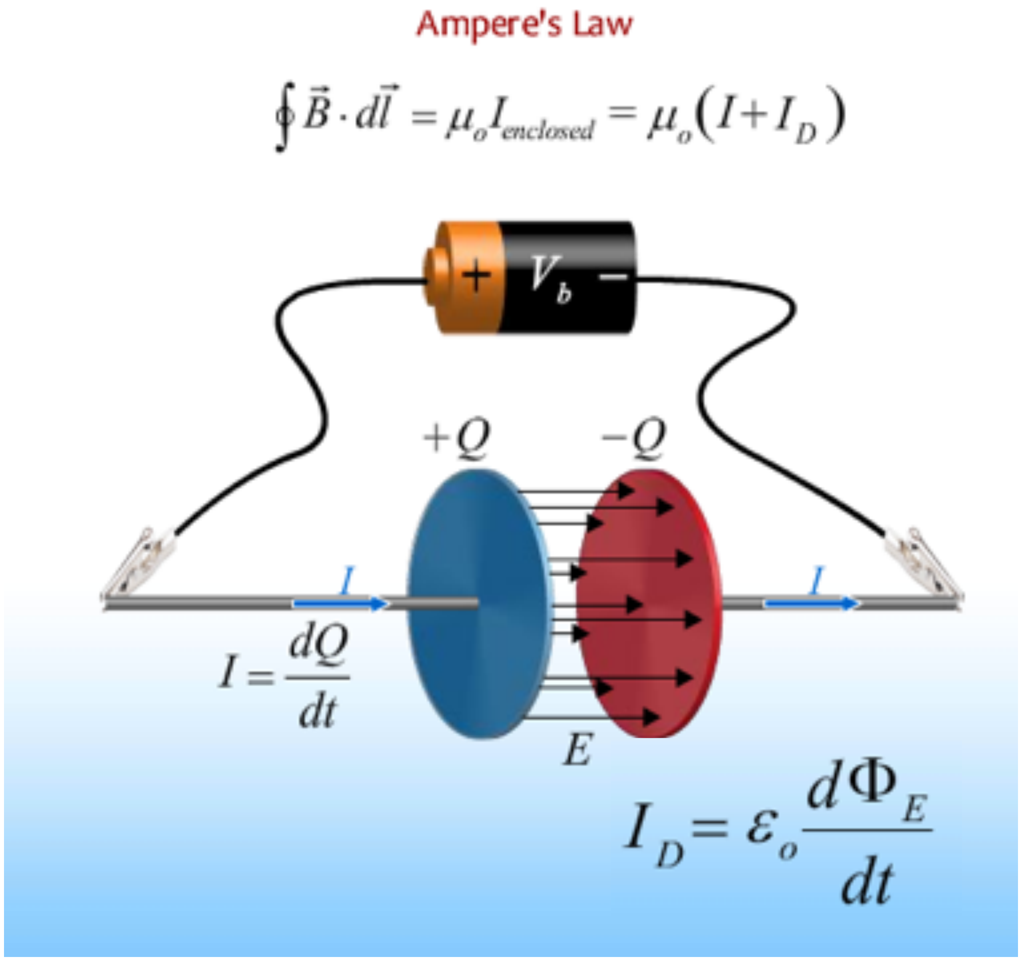
Faraday's Law

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\int \vec{B} \cdot d\vec{A}$$

Ampere's Law

$$\oint \vec{B} \cdot d\vec{l} = \mu_o I_{enclosed}$$

After Prelecture 21: Modify Ampere's Law



$$E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A}$$

$$\downarrow$$

$$\Phi_E = EA = \frac{Q}{\varepsilon_0}$$

$$\downarrow$$

$$Q = \varepsilon_0 \Phi_E$$

$$\downarrow$$

$$\frac{dQ}{dt} = \varepsilon_0 \frac{d\Phi_E}{dt} \equiv I_D$$

Displacement Current

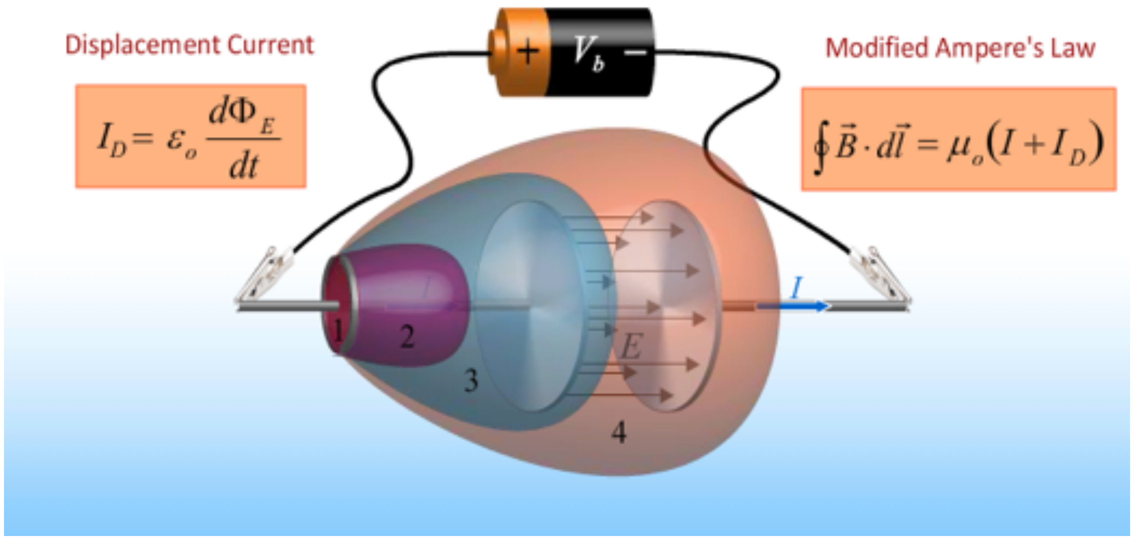
Real Current: Charge Q passes through area A in time t:

$$I = \frac{dQ}{dt}$$

Displacement Current: Electric flux through area A changes in time

$$I_D = \varepsilon_0 \frac{d\Phi_E}{dt}$$

DISPLACEMENT CURRENT and EM WAVES



Faraday's Law

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{A}$$

Modified Ampere's Law

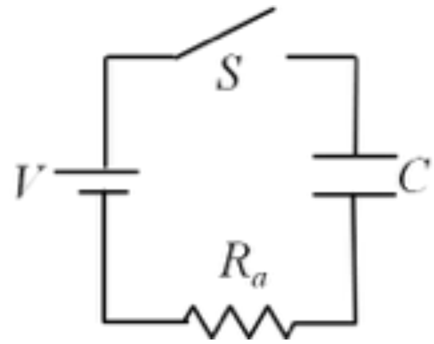
$$\oint \vec{B}\cdot d\vec{l} = \mu_o \varepsilon_o \frac{d}{dt}\int \vec{E}\cdot d\vec{A}$$

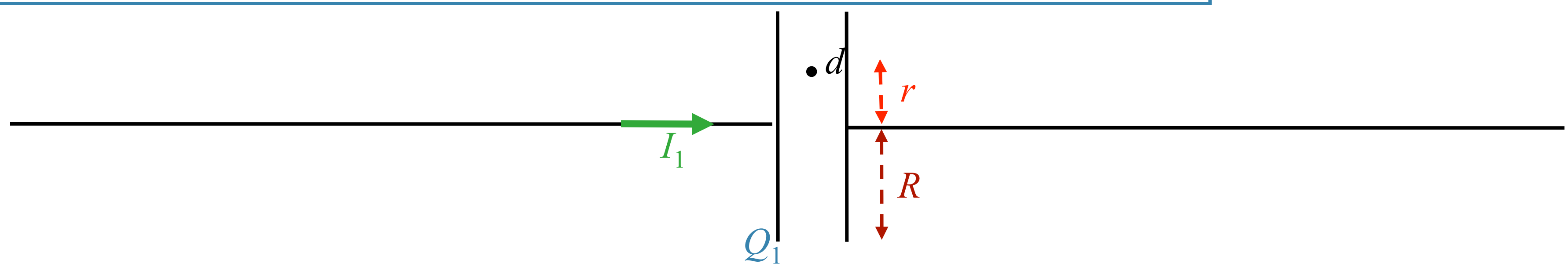
Free space

Calculation

Switch S has been open a long time when at $t = 0$, it is closed. Capacitor C has circular plates of radius R. At time $t = t_1$, a current I_1 flows in the circuit and the capacitor carries charge Q_1.

At time t_1, what is the magnetic field B_1 at a radius r (point d) in between the plates of the capacitor?





Conceptual and Strategic Analysis

Charge Q_1 creates electric field between the plates of C

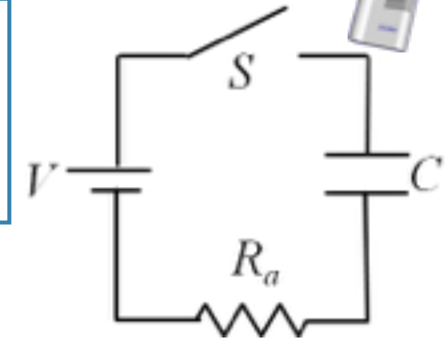
Charge Q_1 changing in time gives rise to a changing electric flux between the plates

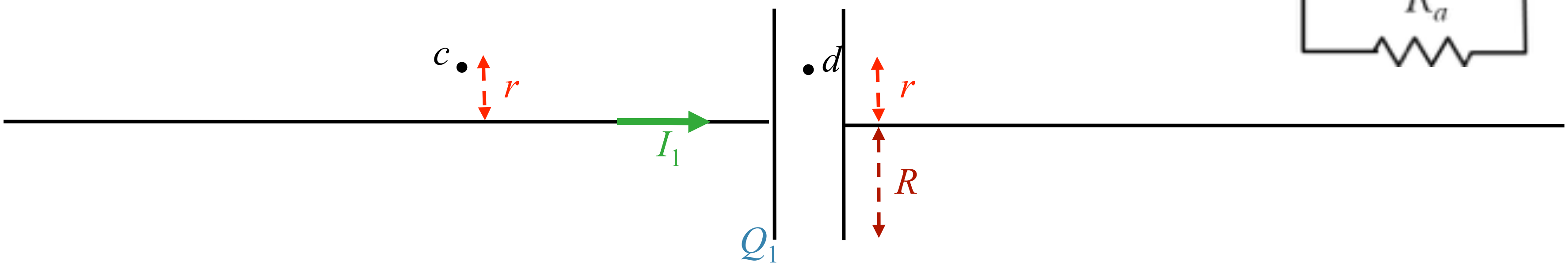
Changing electric flux gives rise to a displacement current I_D in between the plates

Apply (modified) Ampere's law using I_D to determine B

Calculation

Switch S has been open a long time when at $t = 0$, it is closed. Capacitor C has circular plates of radius R. At time $t = t_1$, a current I_1 flows in the circuit and the capacitor carries charge Q_1.



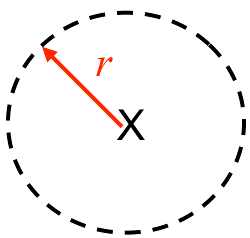


Compare the magnitudes of the B fields at points c and d.

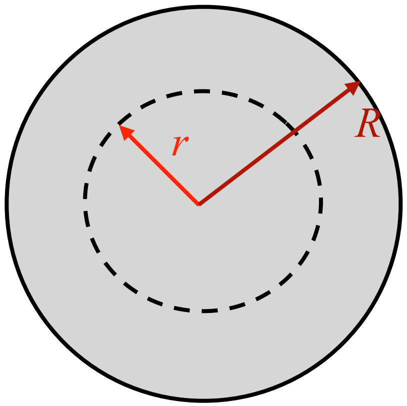
A) $B_c < B_d$ B) $B_c = B_d$ C) $B_c > B_d$

What is the difference?
Apply (modified) Ampere's Law

point c:
I(enclosed) $= I_1$

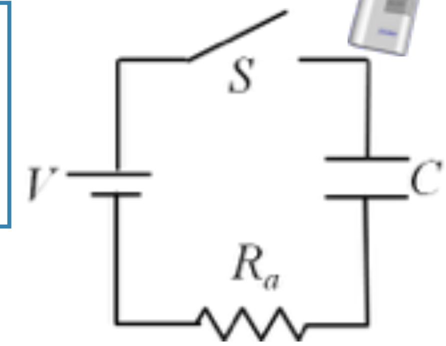


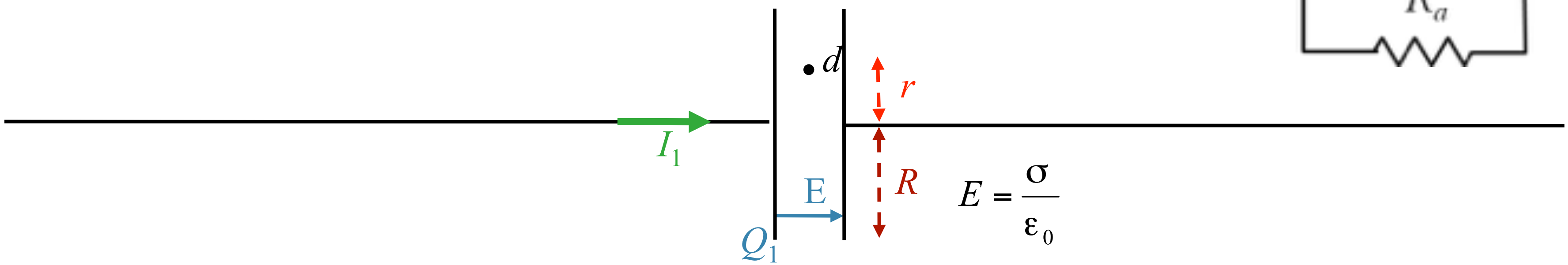
point d:
I_D(enclosed) $< I_1$



Calculation

Switch S has been open a long time when at $t = 0$, it is closed. Capacitor C has circular plates of radius R. At time $t = t_1$, a current I_1 flows in the circuit and the capacitor carries charge Q_1.





$$E = \frac{\sigma}{\varepsilon_0}$$

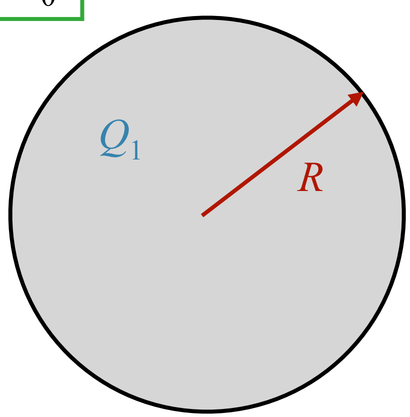
What is the magnitude of the electric field between the plates?

A) $E = \dfrac{Q_1}{\pi R^2 \varepsilon_0}$

B) $E = \dfrac{Q_1 \pi R^2}{\varepsilon_0}$

C) $E = \dfrac{Q_1}{\varepsilon_0}$

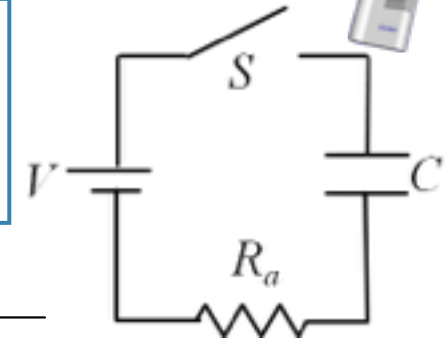
D) $E = \dfrac{Q_1}{r}$

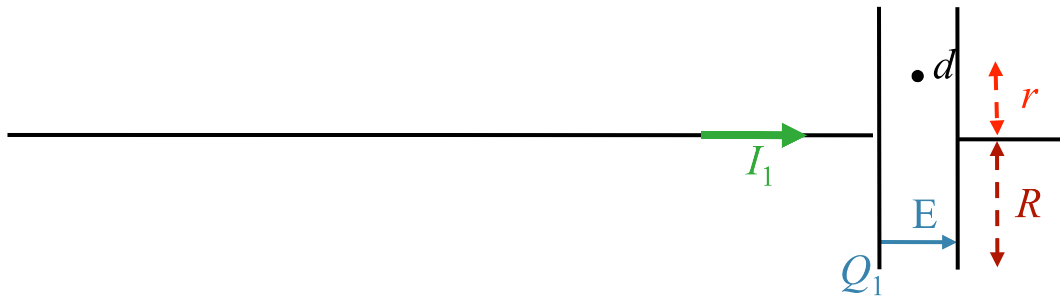


$$E = \frac{\sigma}{\varepsilon_0} \rightarrow \sigma = \frac{Q_1}{A} = \frac{Q_1}{\pi R^2} \rightarrow E = \frac{Q_1}{\varepsilon_0 \pi R^2}$$

Calculation

Switch S has been open a long time when at $t = 0$, it is closed. Capacitor C has circular plates of radius R. At time $t = t_1$, a current I_1 flows in the circuit and the capacitor carries charge Q_1.





$$E = \frac{Q_1}{\pi R^2 \varepsilon_0}$$

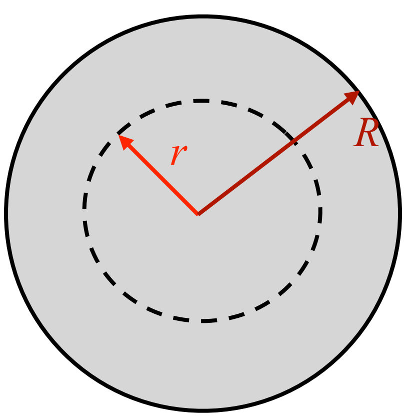
What is the electric flux through a circle of radius r in between the plates?

A) $\Phi_E = \frac{Q_1}{\varepsilon_0} \pi r^2$

B) $\Phi_E = \frac{Q_1}{\varepsilon_0} \pi R^2$

C) $\Phi_E = \frac{Q_1 r^2}{\varepsilon_0 R^2}$

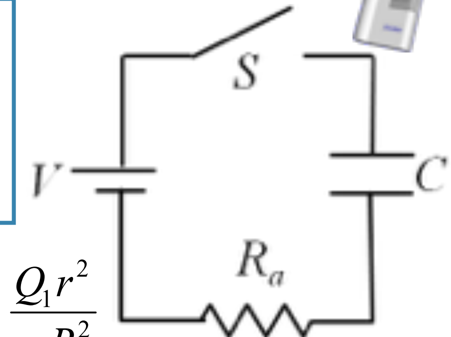
D) $\Phi_E = \frac{Q_1 \pi r^2}{\varepsilon_0 R^2}$



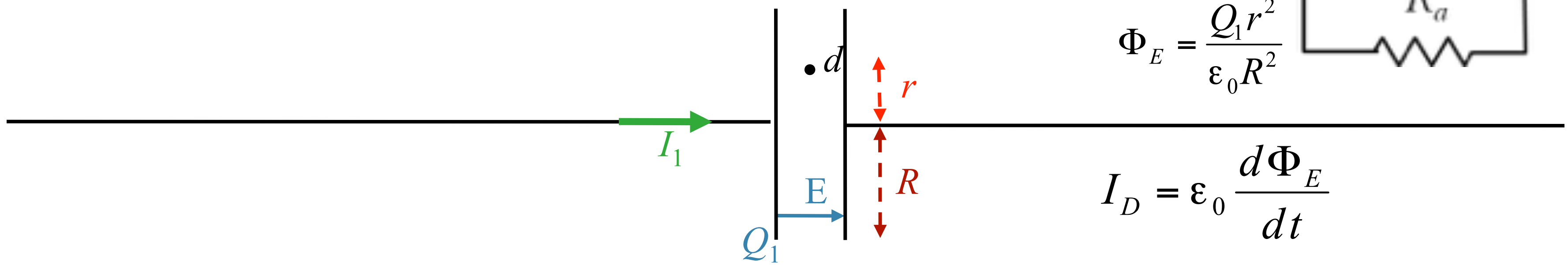
$$\Phi_E = \vec{E} \cdot \vec{A} \rightarrow \Phi_E = \frac{Q_1}{\varepsilon_0 \pi R^2} \pi r^2 \rightarrow \Phi_E = \frac{Q_1}{\varepsilon_0} \frac{r^2}{R^2}$$

Calculation

Switch S has been open a long time when at $t = 0$, it is closed. Capacitor C has circular plates of radius R. At time $t = t_1$, a current I_1 flows in the circuit and the capacitor carries charge Q_1.



$$\Phi_E = \frac{Q_1 r^2}{\varepsilon_0 R^2}$$



$$I_D = \varepsilon_0 \frac{d\Phi_E}{dt}$$

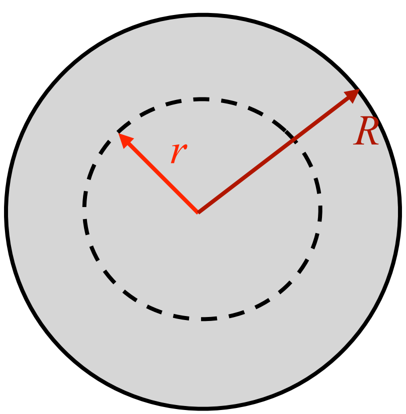
What is the displacement current enclosed by circle of radius r ?

A) $I_D = I_1 \frac{R^2}{r^2}$

B) $I_D = I_1 \frac{r}{R}$

C) $I_D = I_1 \frac{r^2}{R^2}$

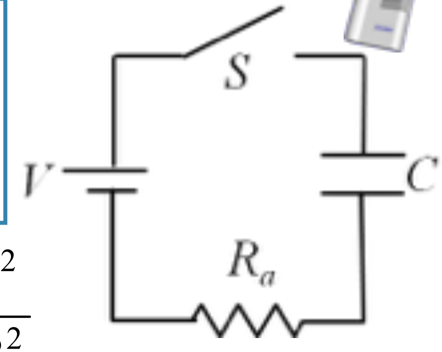
D) $I_D = I_1 \frac{R}{r}$

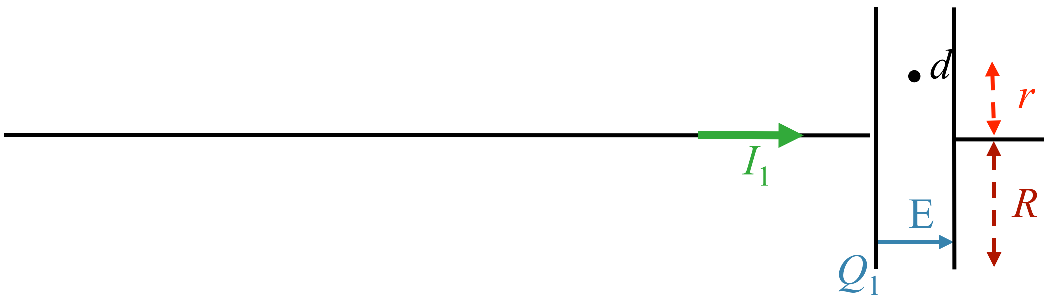


$$I_D = \varepsilon_0 \frac{d\Phi_E}{dt} = \frac{dQ_1}{dt}\frac{r^2}{R^2} = I_1 \frac{r^2}{R^2}$$

$$\rightarrow \quad I_D = I_1 \frac{r^2}{R^2}$$

Switch S has been open a long time when at $t = 0$, it is closed. Capacitor C has circular plates of radius R. At time $t = t_1$, a current I_1 flows in the circuit and the capacitor carries charge Q_1.





$$I_D = I_1 \frac{r^2}{R^2}$$

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 (I + I_D)$$

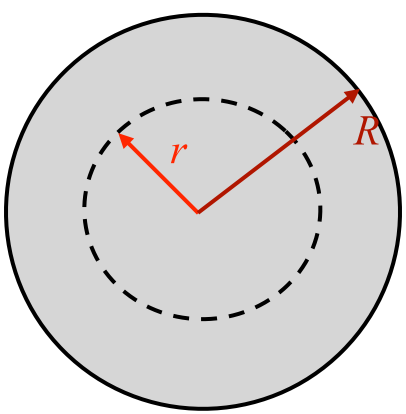
What is the magnitude of the B field at radius r ?

A) $B = \frac{\mu_0 I_1}{2\pi R}$

B) $B = \frac{\mu_0 I_1}{2\pi r}$

C) $B = \frac{\mu_0 I_1}{2\pi} \frac{R}{r^2}$

D) $B = \frac{\mu_0 I_1}{2\pi} \frac{r}{R^2}$



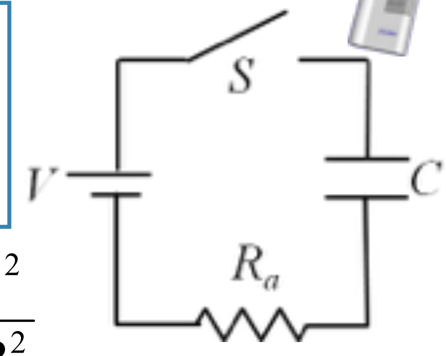
Ampere's Law: $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 (I + I_D)$

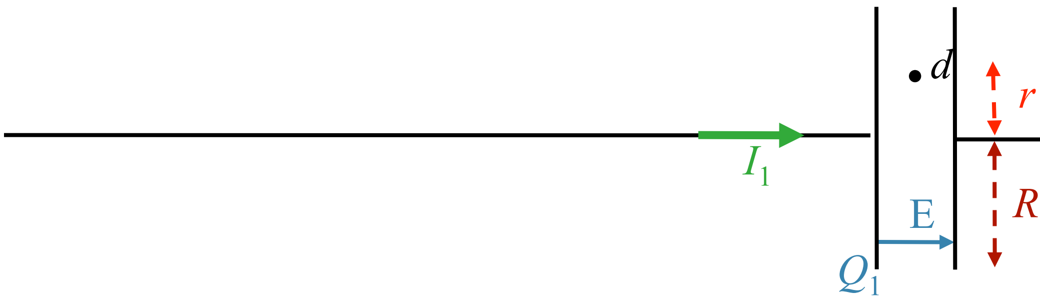
$$\rightarrow \quad B(2\pi r) = \mu_0 \left(0 + I_1 \frac{r^2}{R^2} \right)$$

$$\rightarrow \quad B = \frac{\mu_0 I_1}{2\pi} \frac{r}{R^2}$$

Calculate

Switch S has been open a long time when at $t = 0$, it is closed. Capacitor C has circular plates of radius R. At time $t = t_1$, a current I_1 flows in the circuit and the capacitor carries charge Q_1.



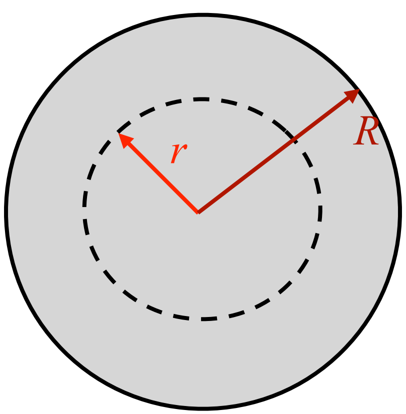


$$I_D = I_1 \frac{r^2}{R^2}$$

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 (I + I_D)$$

What is the magnitude of the B field at radius r ?

$$B = \frac{\mu_0 I_1}{2\pi} \frac{r}{R^2}$$



Let:

$I_1 = 1$ A
$R = 1$ m

What is B at $r = 0.5$ m ?
(answer on next page)

answer

Let:

$I_1 = 1$ A
$R = 1$ m

What is B at $r = 0.5$ m ?

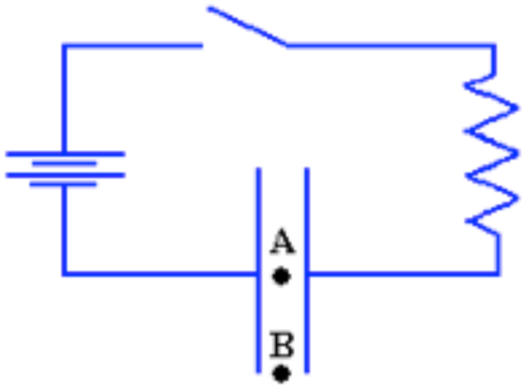
$$B = \frac{\mu_0 I_1}{2\pi} \frac{r}{R^2}$$

$B = (2 \text{ x } 10^{-7})(1)(0.5)/1^2$

$B = 1 \text{ x } 10^{-7}$ T

CheckPoint 2

2) At time t = 0 the switch in the circuit shown below is closed. Points A and B lie inside the capacitor; A is at the center and B is toward the outer edge.



After the switch is closed, there will be a magnetic field at point A which is proportional to the current in the circuit.

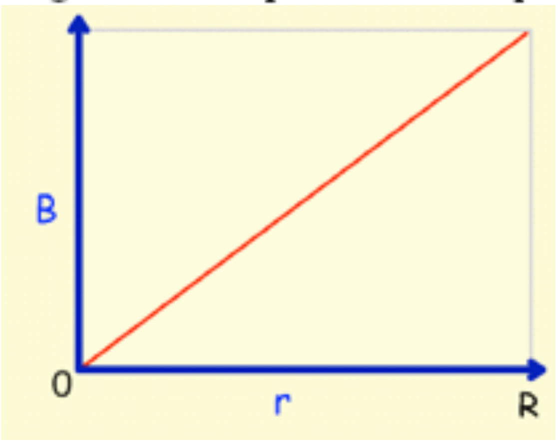
- True
- False

$$B = \frac{\mu_0 I_1}{2\pi}\frac{r}{R^2}$$

B is proportional to *I*

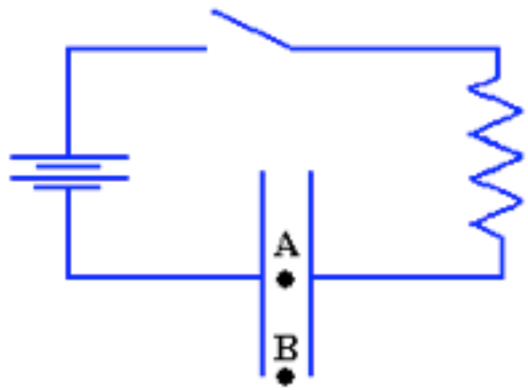
but

At *A*, $B = 0$!!



CheckPoint 4

At time t = 0 the switch in the circuit shown below is closed. Points A and B lie inside the capacitor; A is at the center and B is toward the outer edge.



Compare the magnitudes of the magnetic fields at points A and B just after the switch is closed:

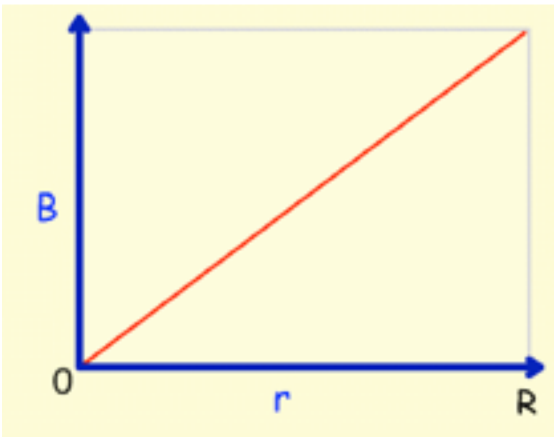
A ○ $B_A < B_B$

B ○ $B_A = B_B$

C ○ $B_A > B_B$

From the calculation we just did:

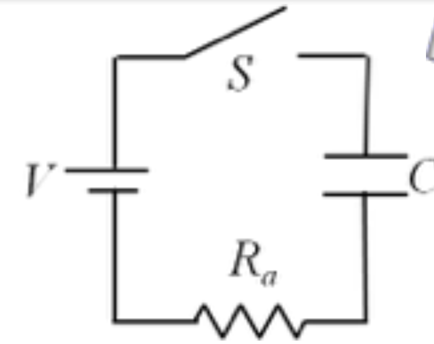
$$B = \frac{\mu_0 I_1}{2\pi} \frac{r}{R^2}$$



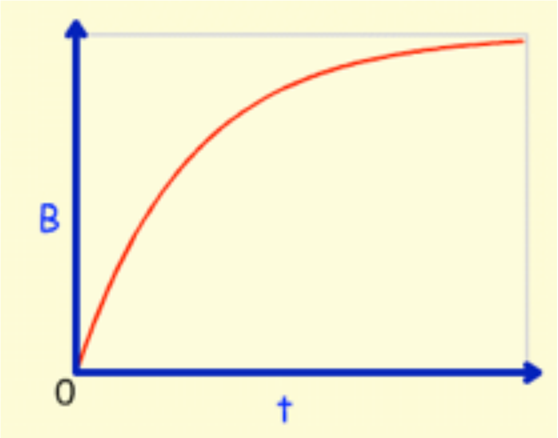
Follow-Up

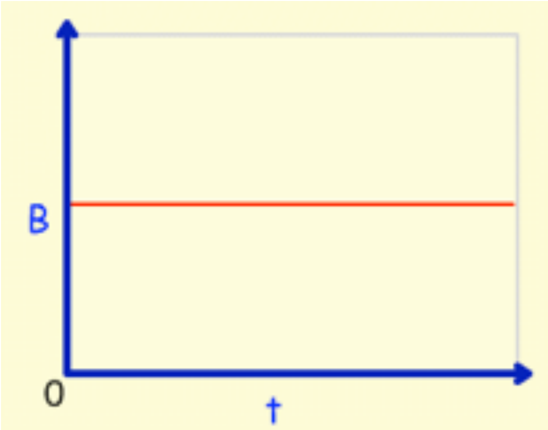
Switch S has been open a long time when at $t = 0$, it is closed. Capacitor C has circular plates of radius R. At time $t = t_1$, a current I_1 flows in the circuit and the capacitor carries charge Q_1.

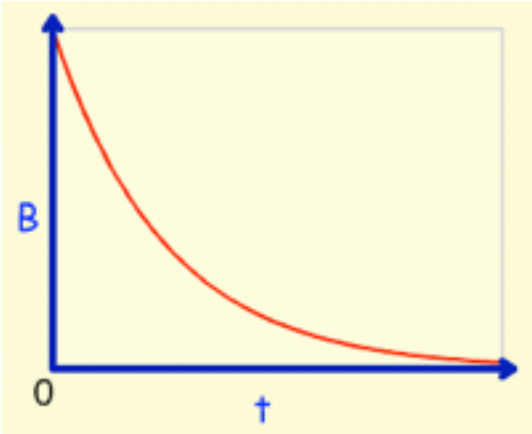
What is the time dependence of the magnetic field B at a radius r between the plates of the capacitor?



$$B_1 = \frac{\mu_0 I_1}{2\pi}\frac{r}{R^2}$$







B at fixed r is proportional to the current I

Close switch: $V_C = 0 \Rightarrow I = V/R_a$ (maximum)

I exponentially decays with time constant $\tau = R_a C$

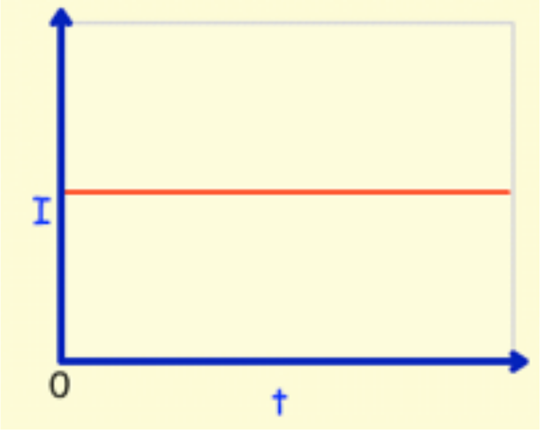
Follow-Up 2

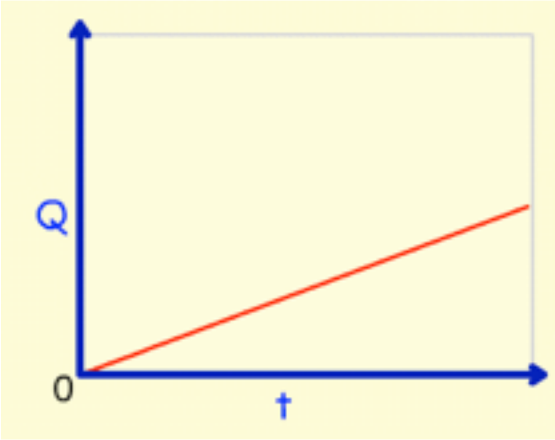
Suppose you were able to charge a capacitor with constant current (*I* does not change in time).

Does a B field exist in between the plates of the capacitor?

A) YES

B) NO





Constant current $\Rightarrow Q$ increases linearly with time

Therefore E increases linearly with time, $E = Q/(A\varepsilon_0)$

dE/dt is not zero $\Rightarrow$ Displacement current is not zero

$\Rightarrow B$ is not zero !

Waves

1-D Wave Equation

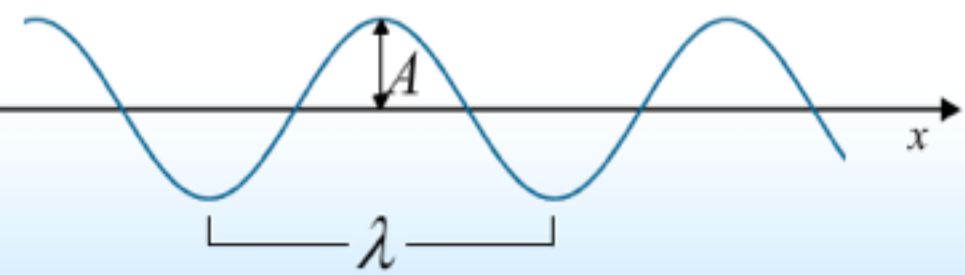
$$\frac{d^2h}{dx^2} = \frac{1}{v^2}\frac{d^2h}{dt^2}$$

Solution

$$h(x,t) = h_1(x - vt) + h_2(x + vt)$$

Common Example: Harmonic Plane Wave

$$h(x,t) = A\cos(kx - \omega t)$$



Variable Definitions

Amplitude: A

Wave Number: $k = \frac{2\pi}{\lambda}$

Wavelength: λ

Angular Frequency: $\omega = \frac{2\pi}{T}$

Period: T

Frequency: $f = \frac{1}{T}$

Velocity: $v = \lambda f = \frac{\omega}{k}$

Faraday's Law

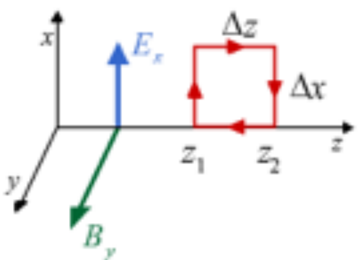
$$\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{A}$$

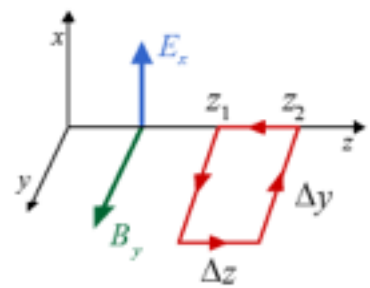
Plane Wave Solution

$$\vec{E} \rightarrow \vec{E}(z,t)$$
$$\vec{B} \rightarrow \vec{B}(z,t)$$

Modified Ampere's Law

$$\oint \vec{B}\cdot d\vec{l} = \mu_o \varepsilon_o \frac{d}{dt}\int \vec{E}\cdot d\vec{A}$$





$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}$$

$$\frac{\partial^2 E_x}{\partial z^2} = -\frac{\partial}{\partial z}\frac{\partial B_y}{\partial t}$$

$$\frac{\partial B_y}{\partial z} = -\mu_o \varepsilon_o \frac{\partial E_x}{\partial t}$$

$$\frac{\partial}{\partial t}\frac{\partial B_y}{\partial z} = -\mu_o \varepsilon_o \frac{\partial^2 E_x}{\partial t^2}$$

$$\frac{\partial^2 E_x}{\partial z^2} = \mu_o \varepsilon_o \frac{\partial^2 E_x}{\partial t^2}$$

Wave Equation

$$\frac{\partial^2 E_x}{\partial z^2} = \mu_o \varepsilon_o \frac{\partial^2 E_x}{\partial t^2}$$

Speed of Electromagnetic Wave

$$v = \frac{1}{\sqrt{\mu_o \varepsilon_o}} = c = 3.00\times 10^8 \text{ m/s}$$

Speed of Light !

Special Relativity (1905)

Speed of Light is Constant

Albert Einstein

"How can light move at the same velocity in any inertial frame of reference? That's really trippy. "

see PHYS 285

$$\frac{\partial^2 E_x}{\partial z^2} = \mu_o \varepsilon_o \frac{\partial^2 E_x}{\partial t^2}$$

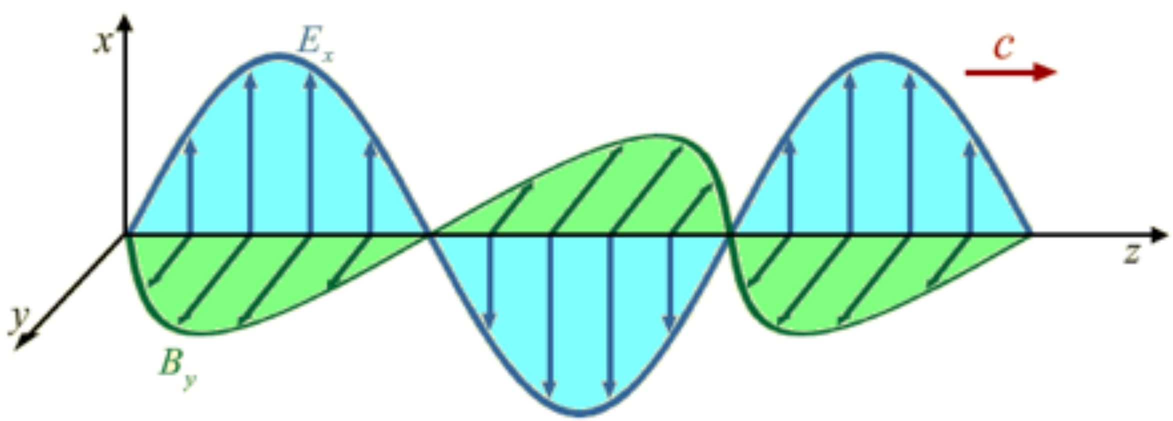
$$\frac{\partial^2 B_y}{\partial z^2} = \mu_o \varepsilon_o \frac{\partial^2 B_y}{\partial t^2}$$

Example: A Harmonic Solution

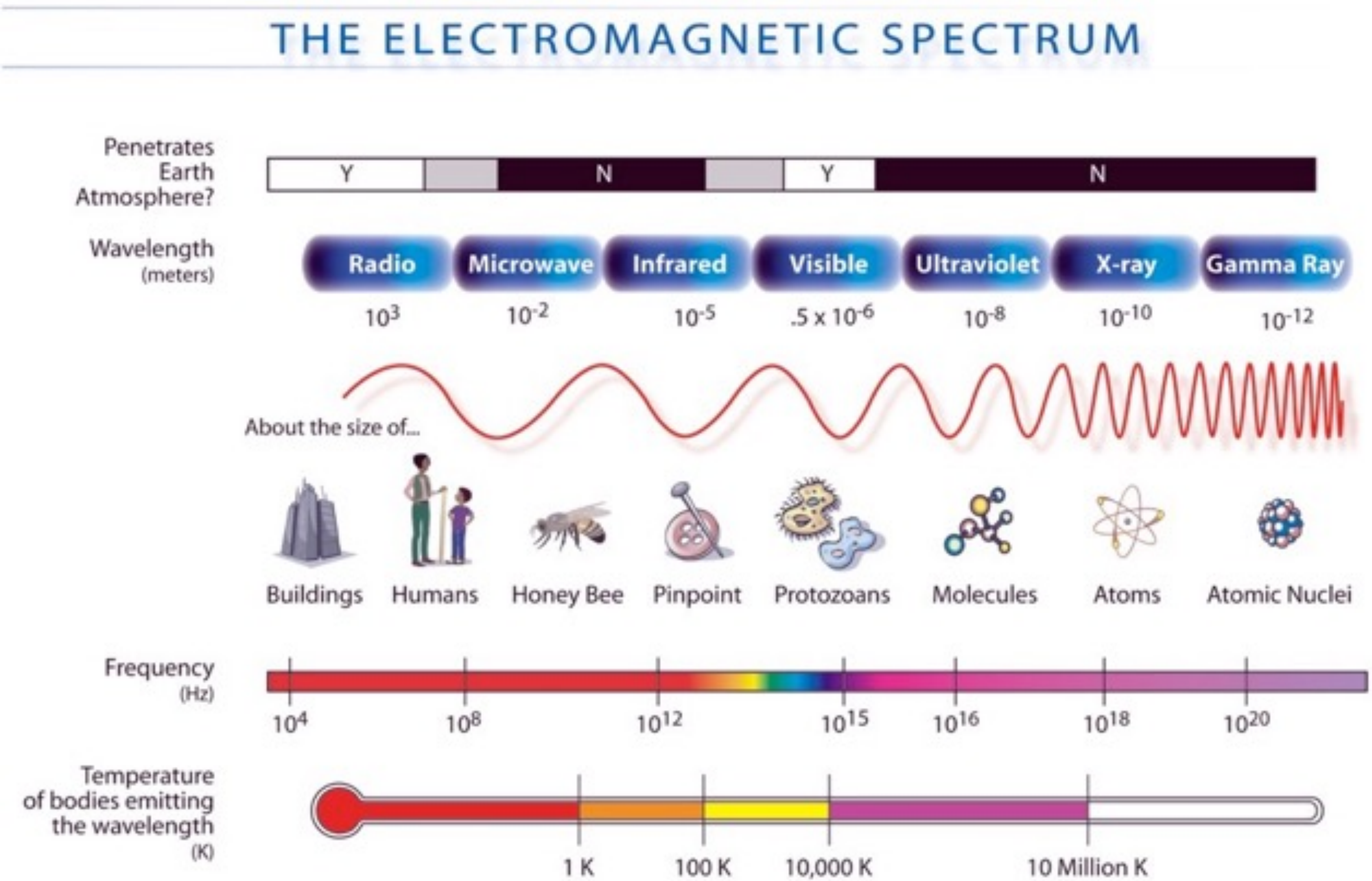
$$E_x = E_o \cos(kz - \omega t) \xrightarrow{\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}} B_y = \frac{k}{\omega} E_o \cos(kz - \omega t)$$

Two Important Features

1. B_y is in phase with E_x
2. $B_o = \frac{E_o}{c}$

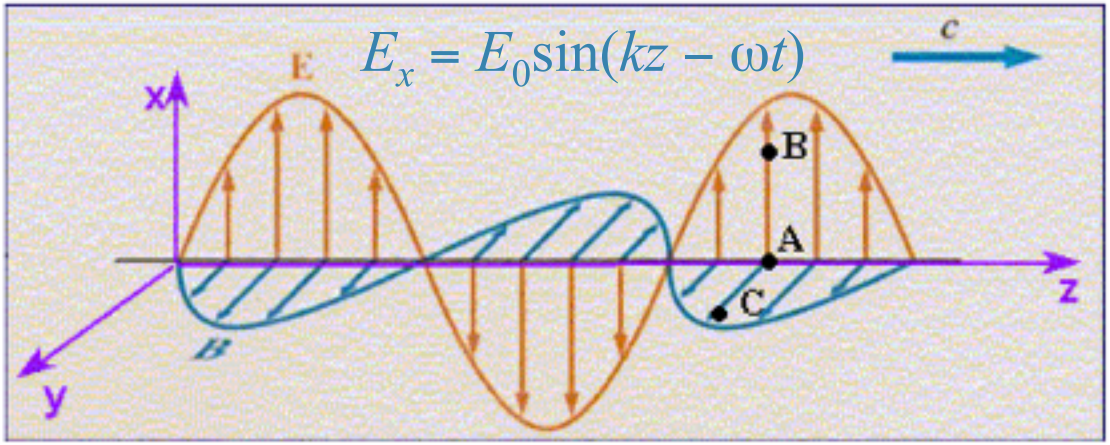


Wave simulations


THE ELECTROMAGNETIC SPECTRUM
Penetrates Earth Atmosphere?
Y
N
Y
N
Wavelength (meters)
Radio
Microwave
Infrared
Visible
Ultraviolet
X-ray
Gamma Ray
10^3
10^-2
10^-5
.5 x 10^-6
10^-8
10^-10
10^-12
About the size of...
Buildings
Humans
Honey Bee
Pinpoint
Protozoans
Molecules
Atoms
Atomic Nuclei
Frequency (Hz)
10^4
10^8
10^12
10^15
10^16
10^18
10^20
Temperature of bodies emitting the wavelength (K)
1 K
100 K
10,000 K
10 Million K

CheckPoint 6

6) An electromagnetic plane-wave is traveling in the +z direction. The illustration below shows this wave an some instant in time. Points A, B, and C have the same z coordinate.



Compare the magnitudes of the electric field at points A and B.

- ○ $E_a < E_b$
- ○ $E_a = E_b$
- ○ $E_a > E_b$

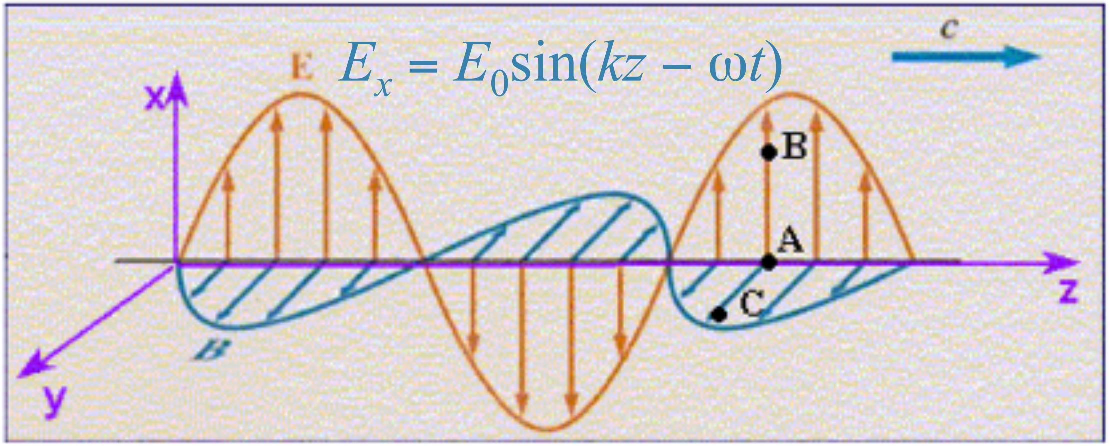
$E = E_0 \sin(kz - \omega t)$:

E depends only on z coordinate for constant t.

z coordinate is same for A, B, C.

CheckPoint 7

An electromagnetic plane-wave is traveling in the +z direction. The illustration below shows this wave an some instant in time. Points A, B, and C have the same z coordinate.



Compare the magnitudes of the electric field at points A and C.

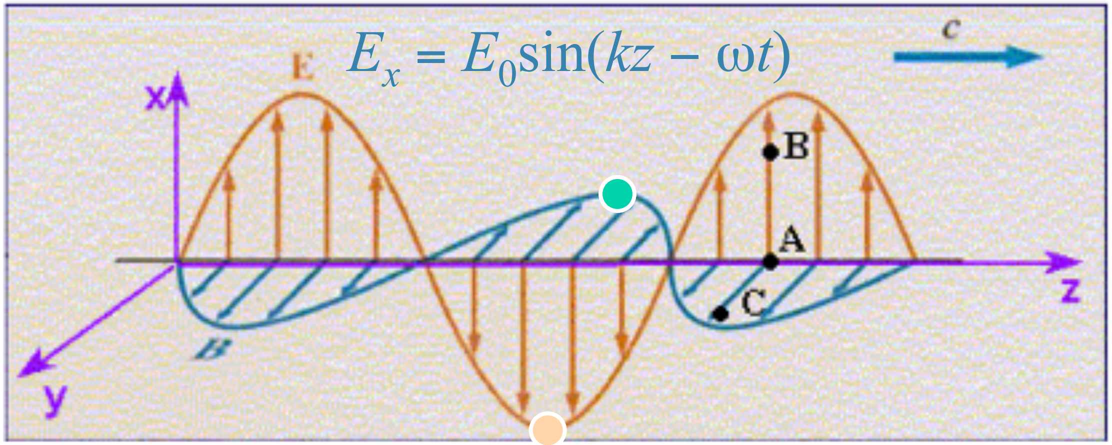
- $E_a < E_c$
- $E_a = E_c$
- $E_a > E_c$

$E = E_0 \sin(kz - \omega t)$:

E depends only on z coordinate for constant t.

z coordinate is same for A, B, C.

Clicker Question



Consider a point (x,y,z) at time t when E_x is negative and has its maximum magnitude.

At (x,y,z) at time t, what is B_y?

A) B_y is positive and has its maximum magnitude

B) B_y is negative and has its maximum magnitude

C) B_y is zero

D) We do not have enough information