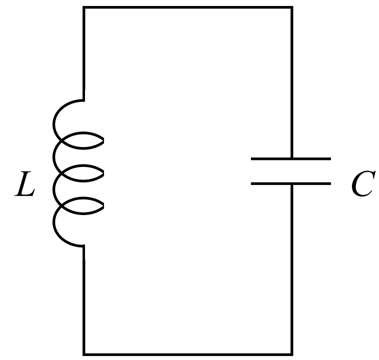
**Discussion Question 11B**
**Physics 212 week 11**
***LC Circuits***




At right, we see the classic **LC circuit**, consisting of an inductor in series with a capacitor. To be precise, this is an **undriven** LC circuit, since there is no battery driving the flow of current. Nevertheless, this simple circuit has an amazing and highly useful property: it supports a resonant oscillation of current. In the ideal case shown here where there is *no* resistance in the circuit, the oscillation can continue *indefinitely* without any external source of EMF to drive it. LC circuits often appear as parts of larger networks which are designed to operate at a particular frequency. The most familiar example is a radio, whose circuitry can be tuned to receive (i.e. respond to) incoming electromagnetic waves of a particular frequency.

$L$ = 300 mH
$C$ = 0.025 μF

**(a) What is the angular frequency $\omega_0$ of the current maintained by the circuit? What is the linear frequency $f_0$?**

An *undriven* LC circuit can *only* operate in a stable way at its natural resonant frequency. As for linear versus angular frequency, the conversion is easy if you remember the units: $f$ is in Hz = 1/sec, $\omega$ is in rad/sec.

The natural frequency is when the net reactance is 0 or $\omega_0 L - \frac{1}{\omega_0 C} = 0 \rightarrow$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.3\left(0.025\times 10^{-6}\right)}} = 11.55 \text{ krad/sec ;}$$

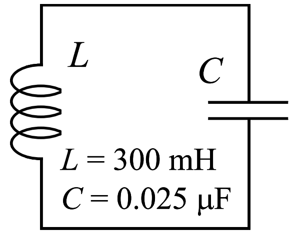
$$\omega_0 = 2\pi f_0 \rightarrow f_0 = \frac{\omega_0}{2\pi} = \frac{11.55 \text{ krad/sec}}{6.28318} = 1838 \text{ Hz}$$

**(b) Let's set our clock so that the current in the circuit is at its maximum value $I_{max}$ at time $t = 0$. Write down an expression for the time-dependence of the current $I(t)$ in terms of $I_{max}$ and the frequency $\omega_0$.**

It's either a sine or a cosine …

We are looking for a trig function that maximizes as t = 0. Hence $I = I_{max} \cos \omega_0 t$

(c) Starting from your expression for $I(t)$, determine the voltages $V_L(t)$ and $V_C(t)$ across the inductor and capacitor. As part of your solution, find the peak values $V_{L,\max}$ and $V_{C,\max}$ in terms of $I_{\max}$.



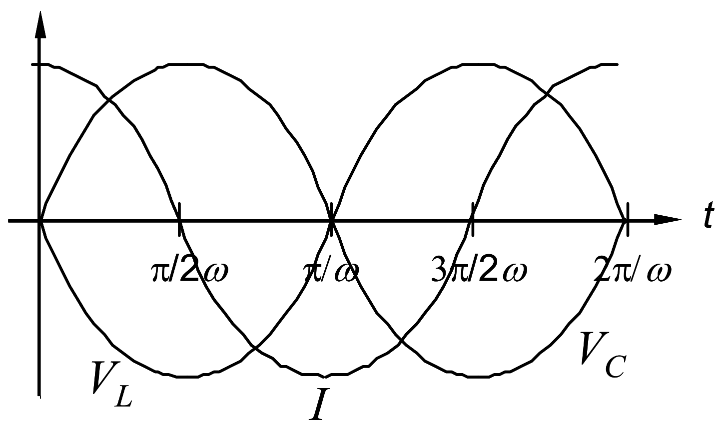

You'll need your familiar formulas for the voltage across $R$, $C$, and $L$ ... and remember that, by definition, $I = \frac{dQ}{dt}$ and so $Q = \int I\,dt$.

$$V_L = L\frac{dI}{dt} = L\frac{d}{dt}\left(I_{\max}\cos\omega t\right) = LI_{\max}\frac{d\omega t}{dt}\left(\frac{d\cos\omega t}{d\omega t}\right) = -\omega L I_{\max}\sin\omega t \rightarrow V_{L,max} = \omega L I_{\max}$$

$V_C = Q/C$ If the current is positive in the clockwise direction, $I = \frac{dQ}{dt}$

$$Q = \int dt\ \mathrm{I} = \int dt\ I_{\max}\cos\omega t = \frac{I_{max}}{\omega}\sin\omega t \rightarrow V_C = \frac{I_{max}}{\omega C}\sin\omega t \rightarrow V_{C,\max} = \frac{I_{max}}{\omega C}$$

**(d) To visualize what's going on, sketch your functions $I(t)$, $V_L(t)$, and $V_C(t)$.** Don't worry about the amplitudes of your curves, just their shapes and phases.




**e) Have a look at your plot → does $V_L$ lead or lag the current? How about $V_C$?**

Leading and lagging can be tricky concepts. Think of it this way: which one "gets there" (i.e. reaches its maximum value) first, $V$ or $I$? The one that "gets there first" leads the other.

$V_L$ leads. $V_C$ lags.

Another way of viewing leading and lagging is if

$I \propto \cos\omega t$ and $V \propto \cos(\omega t + \delta)$, we say V leads I by $\delta$

We can do this using $\cos(\omega t + \delta) = \cos\omega t \, \cos\delta - \sin\omega t \, \sin\delta$

$V_L \propto -\sin\omega t = \cos\left(\omega t + \frac{\pi}{2}\right)$ hence $V_L$ leads I by $90^0$

$V_C \propto \sin\omega t = \cos\left(\omega t - \frac{\pi}{2}\right)$ hence $V_C$ lags I by $90^0$

Congratulations! You've just derived the **master relations** between current and voltage for inductors and capacitors in an AC circuit!

**Peak Values**

$V_{R,\max} = I_{\max} R$

$V_{L,\max} = I_{\max} X_L \rightarrow X_L = \omega L$

$V_{C,\max} = I_{\max} X_C \rightarrow X_C = 1/\omega C$

***Relative Phases***

Across R $\rightarrow$ $V$ in phase with $I$

Across L $\rightarrow$ $V$ leads $I$ by 90°

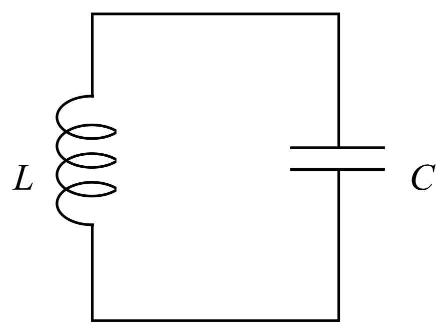
Across C $\rightarrow$ $V$ lags $I$ by 90°

Notice how all the peak-value formulas look like good old "$V = IR$" $\rightarrow$ the **reactances** $X_L$ and $X_C$ describe the <u>effective resistance</u> of inductors and capacitors in AC circuits.

We now set the circuit into oscillation by "stimulating" it with a brief pulse from some external source of EMF. The result is that the peak voltage across the capacitor is $V_{C,\max}$ = 120 V.

**(f) What is the peak current $I_{\max}$?**

$L$ = 300 mH
$C$ = 0.025 μF
$V_{C,\max}$ = 120 V




From part (a) $\omega_0 = \frac{1}{\sqrt{LC}} = 11.5$ krad/sec

From part (c) $V_{C,\max} = \frac{I_{\max}}{\omega_0 C} \rightarrow I_{\max} = \omega_0 C \, V_{C,\max}$

$= (11.5\times10^3 \text{ rad/sec})(0.025\times10^{-6} \text{ F})(120 \text{ V}) = 34.5 \text{ mA}$

**(g) What are the maximum**

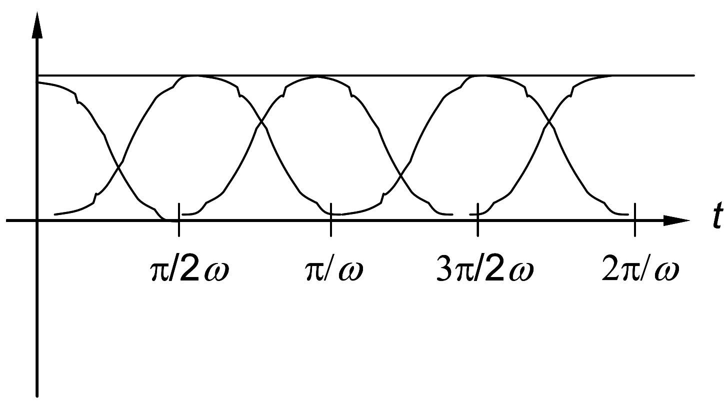
**energies $U_{L,\max}$ and $U_{C,\max}$ stored in the inductor and capacitor respectively?**

$U_{L,max}$ = 1/2 L $I_{max}^2$ = 0.5*(0.3)*($0.0345^2$) = 0.18 mJ

$U_{C,max}$ = 1/2 C $(V_{C,max})^2$ = 0.5*($0.025\times10^{-6}$)*($120^2$) = 0.18 mJ

**(h) Determine time-dependent expressions $U_L(t)$ and $U_C(t)$ for the two stored energies, and add them together to find the total stored energy $U(t)$. Finally, plot your results for all three functions.**

**Something like this:** $U_{total}(t)$ = 0.18 mJ, $U_C$ is a $\sin^2(\omega t)$ function while $U_L$ is a $\cos^2(\omega t)$ function




**(i) How are $U_{L,\max}$, $U_{C,\max}$, and $U_{\max}$ related to each other?**

Knowing this relation is <u>extremely helpful</u> in solving LC circuit problems!

All are equal:

This is no accident but always happens in RC circuits

$$U_{C,max} = \frac{1}{2}C\,V_{C,max}^2 = \frac{1}{2}C\left(X_C I_{max}\right)^2 = \frac{1}{2}C\left(\frac{I_{max}}{\omega C}\right)^2 = \frac{I_{max}^2}{2C}\left(\frac{1}{\omega}\right)^2$$

$$= \frac{I_{max}^2}{2C}\left(\frac{1}{1/\sqrt{LC}}\right)^2 = \frac{I_{max}^2}{2C}LC = \frac{LI_{max}^2}{2} = U_{L,max}$$