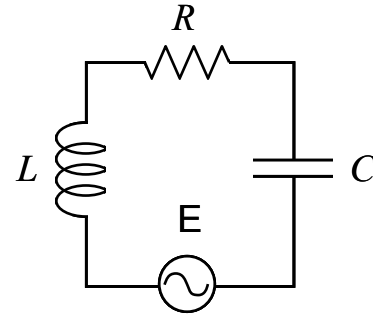


Discussion Question 11C
Physics 212 week 11
Phasors

Next, we encounter the rich subject of **driven RLC circuits**. The most basic example is shown in the diagram: a resistor, a capacitor, and inductor, and an AC generator all connected in series. The **generator** is just a fancy type of battery that produces not a constant “DC” voltage as we have encountered so far, but an oscillating “AC” voltage. The voltage from the generator oscillates sinusoidally at some angular frequency ω , and has a maximum magnitude \mathcal{E}_{max} (also called the “amplitude” or “**peak voltage**”).

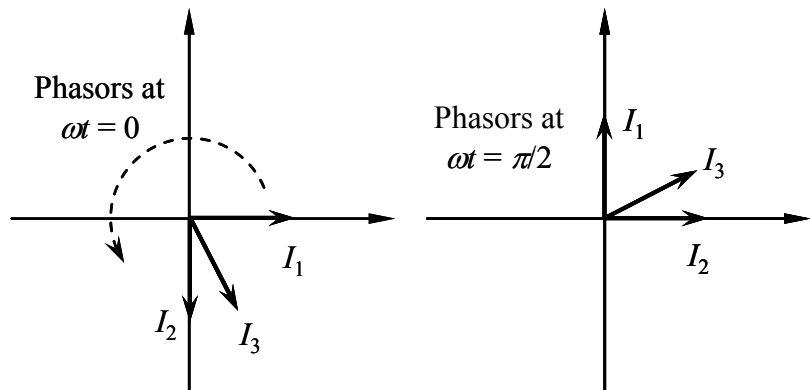


If the AC generator weren't present, our circuit would be a plain old undriven LC circuit with some resistance R thrown in. It would support a nice oscillating current of frequency $\omega_0 = 1/\sqrt{LC}$... except the resistor would damp out the oscillations over time. To keep the circuit going, we attach the generator to **drive** the oscillations. *But ...* the AC generator is driving the circuit at its *own* frequency ω , which need *not* be the same as the circuit's **resonant frequency** ω_0 . The result is something of a mess. ☹ The generator forces the current to oscillate at frequency ω , but if ω doesn't match ω_0 , the driving voltage will be **out of phase** with the current.

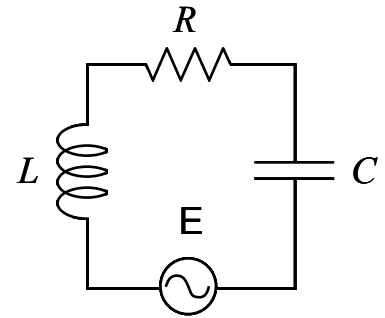
To help us visualize what is going on, we use **phasors**. A phasor is just a way of graphically representing the time-dependence of something which oscillates. Specifically, a phasor is a vector in the xy plane. We then imagine that this vector *rotates* around the origin in a counterclockwise direction, with angular velocity ω radians/second. The phasor describes the behavior of *something* oscillating with frequency ω in the following way: the value of that something, at any given point in time, is equal to the projection of the phasor onto the vertical axis. All clear? OK. Phasors are just a graphical trick to help you figure out how oscillating things behave. Now let's see how to apply them to a driven RLC circuit.

(a) Let's practice by drawing a few phasors. Below are expressions describing three time-dependent currents. Draw the phasors for these currents at two different times: $\omega t = 0$ and $\omega t = \pi/2$. Note that the length of the phasor never changes. What does it represent?

- $I_1(t) = I_{1,\text{max}} \sin(\omega t)$
- $I_2(t) = -I_{2,\text{max}} \cos(\omega t)$
- $I_3(t) = I_{3,\text{max}} \sin(\omega t - \pi/3)$



Now consider the RLC circuit shown. The values of R , L , and C are all known. We also know that the generator is driving the circuit at frequency $f = 50$ Hz and that the peak current is $I_{\max} = 0.5$ A. We will set our clock so that the current is zero at time $t = 0$... thus:

$$I(t) = I_{\max} \sin(\omega t)$$


$$\begin{aligned} L &= 40 \text{ mH} & f &= 50 \text{ Hz} \\ C &= 2.5 \text{ mF} & I_{\max} &= 0.5 \text{ A} \\ R &= 15 \, \Omega \end{aligned}$$

(b) Using the “master relations” for peak current and voltage, calculate $V_{R,\max}$, $V_{C,\max}$, and $V_{L,\max}$.

$$\omega = 2\pi (50) = 314.16 \text{ rad/sec}$$

$$V_{R,\max} = I_{\max} R = (0.5)(15) = 7.5 \text{ V}$$

$$V_{L,\max} = I_{\max} \omega L = (0.5)(314.16)(0.04) = 6.28 \text{ V}$$

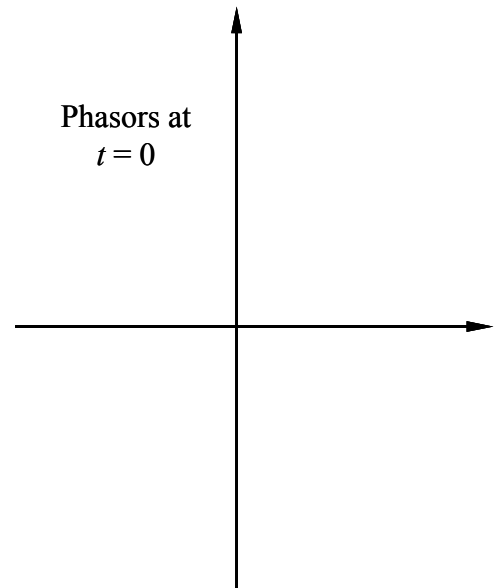
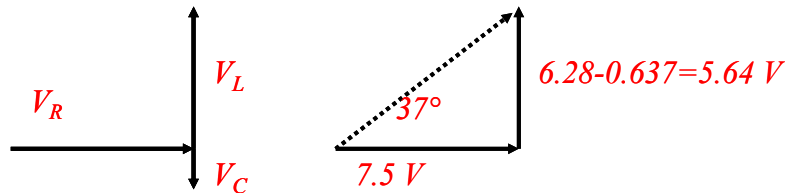
$$V_{C,\max} = I_{\max} / \omega C = 0.5 / (314.16 * 0.0025) = 0.637 \text{ V}$$

(c) Draw a diagram showing the phasors for $I(t)$, $V_R(t)$, $V_C(t)$, and $V_L(t)$ at $t = 0$.

You must use the “master relations” for the phase between current and voltage to determine the angles of the phasors. Also, be sure to draw the *lengths* of your voltage phasors to scale.

$$\tan \phi = (X_L - X_C) / R$$

$$\phi = 37^\circ$$



(d) The phasor diagram makes it easy to answer questions like these:

- True or false? Whenever the voltage across the capacitor hits its maximum value, so does the voltage across the inductor. **False.** When the capacitor voltage hits maximum the inductor voltage hits minimum since they are 180 degrees out of phase.
- True or false? The voltage across the capacitor is always zero when the current through the inductor is zero. **False.** The current is the same through the inductor, capacitor, or

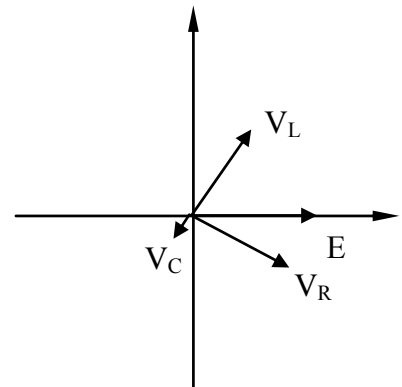
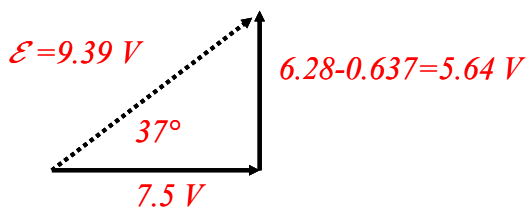
resistor since all elements are in series. The voltage across the capacitor lags the voltage across the resistor (or the current) by 90 degrees and hence the voltage across the capacitor is either a minimum or maximum when the current is zero.

- True or false? When the magnetic energy stored in the inductor hits its maximum value, so does the electric energy stored in the capacitor. **False.** The magnetic energy is proportional to the square of current. The electrical energy is proportional to V_C . The capacitor voltage lags the current by 90 degrees and hence when the current hits a maximum the voltage across the capacitor is zero.

We now know the voltages across the capacitor, resistor, and inductor. The only thing left to calculate is the EMF $\mathcal{E}(t)$ across the generator. From Kirchhoff's Voltage Law, we see immediately that $\mathcal{E}(t) = V_R(t) + V_C(t) + V_L(t)$. To perform the sum, we need one more principle: **Phasors add like vectors.**

(e) What is the voltage \mathcal{E} across the generator? Copy your previous phasor diagram, and add a new phasor representing \mathcal{E} . What is the length of this phasor? What is the angle ϕ that this phasor makes with respect to the current phasor?

$$E = 9.39 \text{ V}, \phi = 37^\circ$$



If you avoided plugging in numbers until the end of your calculation, you just found the last of the “master relations”:

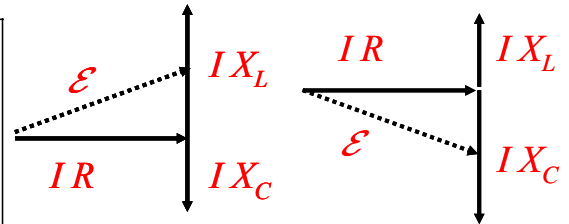
| Peak Values | Relative Phases |
|--|--|
| $E_{\max} = I_{\max} Z \rightarrow Z = \sqrt{R^2 + (X_L - X_C)^2}$ | Across $E \rightarrow V$ leads I by ϕ , with $\tan \phi = (X_L - X_C)/R$ |

These are the current/voltage relationships for the *entire RLC circuit*, as seen by the generator. The **impedance** Z is particularly important: it is the effective resistance of the entire circuit.

(f) What condition would you have to place on the frequency ω to make the EMF \mathcal{E} across the generator lead the current?

Look at your phasor diagram, and remember that adjusting the frequency allows you to change the effective resistances of the inductor and capacitor. The one with the highest reactance will determine the phase of the circuit.

As shown in the two figures, if $X_L > X_C$, \mathcal{E} will lead I whereas if $X_L < X_C$, \mathcal{E} will lag I . $X_L > X_C$ implies $\omega L > \frac{1}{\omega C}$ or $\omega^2 > \frac{1}{LC} \rightarrow \omega > \frac{1}{\sqrt{LC}}$ or $\omega > \omega_0$



(g) What frequency ω would you need to make the peak EMF \mathcal{E}_{\max} as small as possible for a fixed I_{\max} ?

From that last “master relation”, it’s clear that you need to *minimize* the *impedance* = effective total resistance of the circuit ... what is the minimum possible impedance Z ?

Since the reactance leads the resistance by $\pm 90^\circ$ by right triangles we have $Z = \sqrt{R^2 + (X_L - X_C)^2}$. The minimal impedance occurs when $X_L - X_C = 0 \rightarrow \omega L = \frac{1}{\omega C} \rightarrow \omega = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(40 \times 10^{-3})(2.5 \times 10^{-3})}} = 100 \text{ rad/sec}$

(h) The frequency you found in (f) is the magic resonant frequency ω_0 of the circuit. Draw the phasor diagram for the circuit when it is in resonance.

Note how that nasty phase ϕ disappears at resonance. ☺

