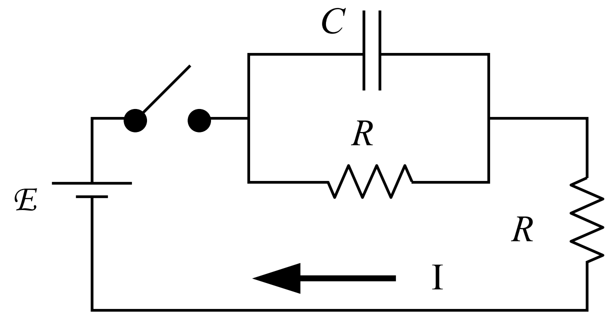
Discussion Question 7C
P212, Week 7
RC Circuits



The circuit shown initially has the capacitor uncharged, and the switch open. At time $t = 0$, the switch is thrown. Write all answers in terms of $\mathcal{E}$, R, and C as needed.

(a) At $t = 0^+$, immediately after the switch is thrown what is the current $I(0^+)$**?**

Consider applying KVL around the loop that includes the battery and the capacitor.
$\mathcal{E}$ - $\Delta V_{cap} - IR = 0$. At t=0^+ Q_{cap}=0 $\rightarrow$ $\Delta V_{cap} = 0 \rightarrow \mathcal{E} - IR = 0 \rightarrow I(0^+) = \mathcal{E}/R$

(b) After a very long time, what is the current I_∞ ?

When the capacitor is fully charged it acts like an open circuit. Hence all current flows through $R_1 + R_2$. The current is then $I_\infty = \mathcal{E}/2R$.

(c) After a very long time, what is the charge on the capacitor Q_∞ ?

Consider the KVL that just consists of the capacitor and the one resistor. $I_\infty R - \Delta V_{cap} = 0$
$\rightarrow I_\infty R - Q_\infty / C = 0 \rightarrow \frac{\mathcal{E}}{2R} R - Q_\infty / C = 0 \rightarrow Q_\infty = \frac{\mathcal{E}}{2} C$

(d) Assume $Q(t) = Q_\infty(1 - \exp(-\beta t))$ gives the charge on the capacitor as a function of time. Use $\left[\frac{dQ}{dt}\right]_{0^+} = I\left(0^+\right)$ and your answers to **(a)** and **(c)** to compute β. Why does $\left[\frac{dQ}{dt}\right]_{0^+} = I\left(0^+\right)$? What does your β imply for the effective resistance that you use in $\tau = R_{\text{effective}}$ C ?

Initially there is no current flow through the resistor in parallel with the capacitor
Hence I(0) flows entirely through the capacitor. The rate of change of charge
is the current hence $\frac{dQ}{dt}\left(0^+\right) = I\left(0^+\right)$. $\left[\frac{dQ}{dt}\right]_0 = Q_\infty\left[\frac{d}{dt}\left(1 - e^{-\beta t}\right)\right]_0 = \beta Q_\infty$

$$\left[\frac{dQ}{dt}\right]_0 = \beta Q_\infty = \beta\left(\frac{C\mathcal{E}}{2}\right) = I\left(0^+\right) = \frac{\mathcal{E}}{R} \rightarrow \beta = \frac{2}{RC} \rightarrow \tau = \frac{RC}{2} \rightarrow R_{\text{effective}} = \frac{R}{2}$$

Essentially the two resistors act as if they were in parallel during the charging operation.
We would get the same time constant if we discharged the capacitor by shorting the battery/

(e) Use KVL for a loop that includes the capacitor and the battery to compute $I(t)$. Check that your answers are consistent with your answers to parts (a) and (b).

$$\mathcal{E}-\Delta V_{\text{cap}}-IR=0\rightarrow\mathcal{E}-\frac{Q}{C}-IR=0\rightarrow I=\frac{1}{R}\left(\mathcal{E}-\frac{Q}{C}\right)\rightarrow I=\frac{1}{R}\left\{\mathcal{E}-\frac{1}{C}\left(\frac{\mathcal{E}C}{2}\left[1-\exp\left(-\frac{2t}{RC}\right)\right]\right)\right\}$$

$$\rightarrow I=\frac{\mathcal{E}}{R}\left(1-\left[\frac{1}{2}-\frac{1}{2}\exp\left(-\frac{2t}{RC}\right)\right]\right)=\frac{\mathcal{E}}{2R}\left(1+\exp\left(-\frac{2t}{RC}\right)\right)$$

At $t=0^{+}, 1+\exp\left(-\frac{2t}{RC}\right)=2\Rightarrow I=\frac{\mathcal{E}}{R}$ which agrees with (a)

As $t\rightarrow\infty, 1+\exp\left(-\frac{2t}{RC}\right)\rightarrow 1\Rightarrow I=\frac{\mathcal{E}}{2R}$ which agrees with (b)