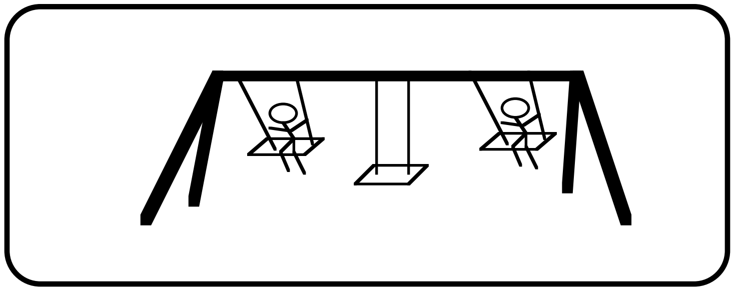
Name ______________________ Date(YY/MM/DD) ______/_________/_______
St.No. __ __ __ __ __-__ __ __ __ Section__________

UNIT 14: HARMONIC MOTION



Back and Forth and Back and Forth. . .
Cameo

OBJECTIVES

1. To learn directly about some of the characteristics of periodic motion – period, frequency, and amplitude.

2. To understand the basic properties of Simple Harmonic Motion (SHM), in which the displacement of a particle varies sinusoidally in time.

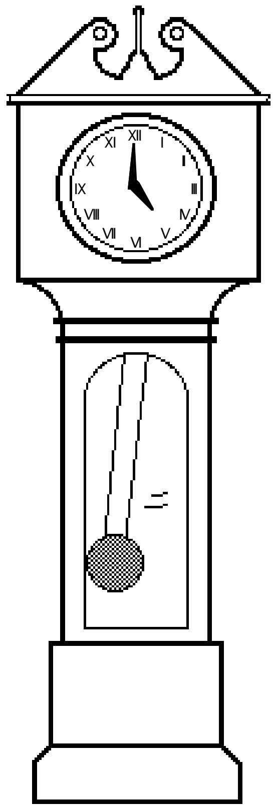
3. To show experimentally that a mass oscillating on a spring undergoes, within the limits of experimental uncertainty, Simple Harmonic Motion.

4. To explore theoretically the factors that influence the rate of oscillation of a mass-spring system using Newton's laws.

5. To explore the harmonic oscillations of the simple pendulum and the relationship between period, mass, and length of the pendulum, both experimentally and theoretically.

 Supported by FIPSE (U.S. Dept. of Ed.) and NSF. Modified for SFU by N. Alberding, 2005

OVERVIEW



Any motion that repeats itself regularly is known as *harmonic* or *periodic* motion. The pendulum in a grandfather clock, molecules in a crystal, the vibrations of a car after it encounters a pothole on the road, and the rotation of the earth around the sun are examples of periodic motion. In this unit we will be especially interested in a type of periodic motion known as Simple Harmonic Motion, which is often called SHM. SHM involves a displacement that changes sinusoidally in time. In this unit, you will explore the mathematical significance of the phrase "displacement that changes sinusoidally in time". You will also study the mathematical behaviour of two classical systems that undergo SHM – the mass on a spring and the simple pendulum.

SHM is so common in the physical world that your understanding of its mathematical description will help you understand such diverse phenomena as the behaviour of the tiniest fundamental particles, how clocks work, and how pulsars emit radio waves. Pendula and masses on springs are merely two examples of thousands of similar periodic systems that oscillate with simple harmonic motion.

In this unit we will study two oscillating systems: a mass and spring system and a simple pendulum. You will need to devise some ways to describe oscillating systems in general and then apply these descriptions to help you observe periodic oscillations. There are several questions you must address in the next three sessions: What is periodic motion and how can it be characterized? What factors does the rate of oscillation of a simple pendulum really depend on? What mathematical behaviour is required of a periodic system to qualify its motion as *harmonic*? Is the oscillation of a mass on the end of a spring really harmonic? How do Newton's laws allow us to predict that the motion of a mass on an ideal spring will be harmonic?

SESSION ONE: OSCILLATING SYSTEMS

Some Characteristics of Oscillating Systems

A mass on a spring, a simple pendulum, and a "particle" rotating on a wheel with uniform angular velocity can be adjusted so they undergo oscillations that have some similarities. To observe these three systems you need the following apparatus:

- A pendulum bob
- A string of variable length
- A spring (mounted with the tapered end up)
- A mass pan and masses
- A rotating disk with a pin on its outer rim
- A variable speed motor to drive the disk
- Clamp stands and rods for support

These three systems are pictured below.

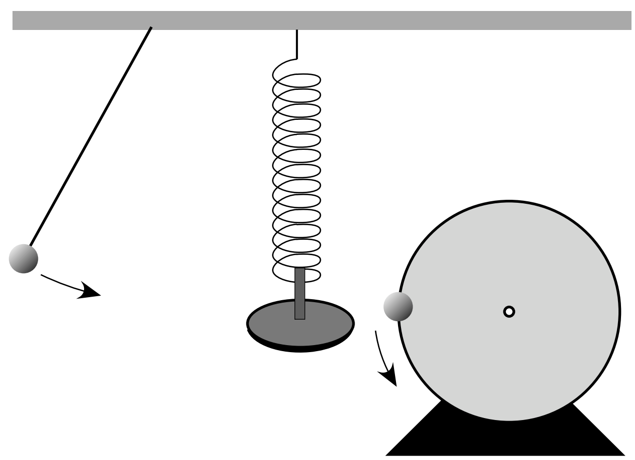


Figure 14-1: The pendulum, spring and mass, and rotating disk as oscillating systems.

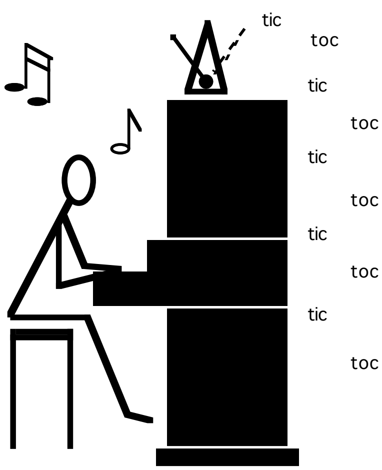
✍ Activity 14-1: Periodic System Similarities

(a) Describe in your own words what characteristic of all three systems seems to be the *same*.

(b) Describe what additional characteristic seems to be the same about the spring-mass system and the rotating disk.

Useful Definitions for Oscillating Systems

In describing the similarities between the oscillating systems it would be useful if we all used a common vocabulary. The three terms used most often in describing oscillations are the following:



Period – The time it takes for the oscillating particle to go through one complete cycle of oscillation. This is commonly denoted by the capital letter T.

Frequency – The number of cycles the oscillating particle makes in one second. This is denoted by the symbol f or the Greek letter small ν (pronounced nu) in most textbooks. The units of frequency are hertz where 1 hertz = 1 cycle/sec. The unit hertz is abbreviated Hz.

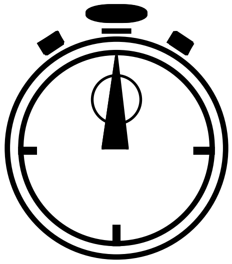
Amplitude – The maximum displacement of the oscillating particle from its equilibrium position. Following the convention in this course, displacement is usually measured in meters. The symbol for amplitude is often the capital letter A. Other letters such as X_{max} or Y_{max} are also used. In the case of the simple pendulum, the amplitude is usually measured in radians and denoted by θ_{max}.

Two of these definitions, *frequency* and *period*, are related to each other. By observing the systems shown in Figure 14-1 above, you should be able to find a mathematical equation that relates the frequency of a given oscillating system to its period. To do the needed observations you'll need the following items:

- a spring-mass system w/ several masses
- a disk rotating with constant angular velocity
- a simple pendulum
- a stop watch

Warning! If you add too much mass to a spring it will become permanently stretched and hence ruined.

✍ Activity 14-2: Relating Period and Frequency



Take a look at the three systems discussed above. Use the stop watch and record the average period and frequency of the object in question in each case.
Hint: For more accuracy, you can count cycles for a long time and divide the total number of cycles by the total time in seconds to get the frequency in hertz. Also, you can time multiple periods and divide the total time by the number of periods.

(a) Amplitude:

Pendulum (rad) ______

Mass on spring (m) ______

Particle on wheel (m) ______

(b) Period (s):

Pendulum ______

Mass on spring ______

Particle on wheel ______

(c) Frequency (Hz):

Pendulum ______

Mass on Spring ______

Particle on wheel ______

(d) Is there any obvious mathematical relationship between the period and frequency? For example, what happens to the period when the frequency doubles? *Use mathematical data to confirm this relationship*, not just qualitative reasoning. To get this data you can vary the amount of mass loaded on a spring.

Period (s)	Frequency (Hz)

Graphing Periodic Motion Using a Motion Detector

What is the nature of the displacement as a function of time for the spring-mass system? Can you predict how the displacement will vary with time? How will your prediction compare with actual observations? This activity involves using a motion detector system to track the displacement of a mass on a spring. For this activity you'll need the following apparatus:

- A spring
- Some masses to hang on the spring
- A clamp stand
- A motion detection system

The setup is shown in the figure below.

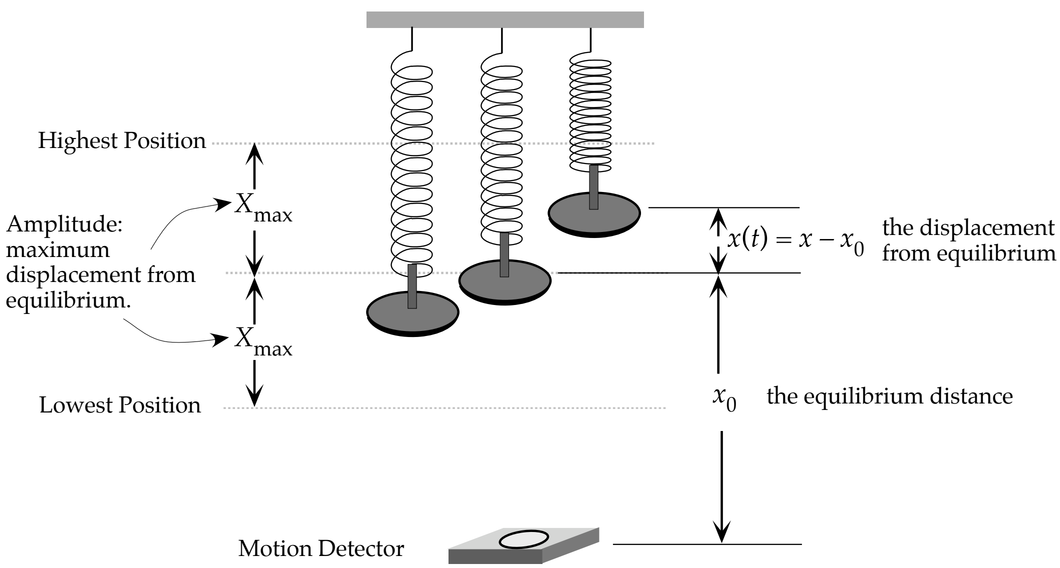
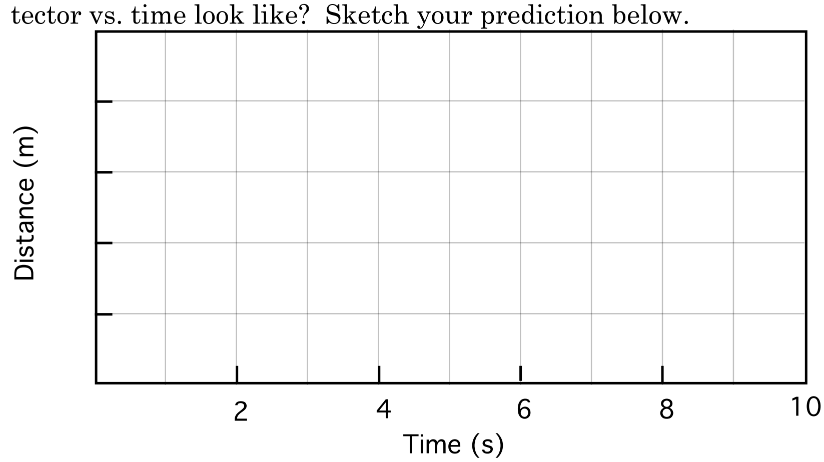


Figure 14-2: Diagram of the setup for the Graphical Observation of the Motion of a Mass on a Spring.

Pull down on your spring to obtain a good healthy amplitude. *(Somewhere between a small displacement and one that stretches the spring so much that it remains permanently distorted.)* Let the mass go. As you watch the mass oscillating on the spring you can see the mass going from a maximum displacement to no displacement and then to a maximum displacement in the opposite direction. What do you expect a graph of this motion to look like?

✍ Activity 14-3: Position Graph for a Mass on a Spring

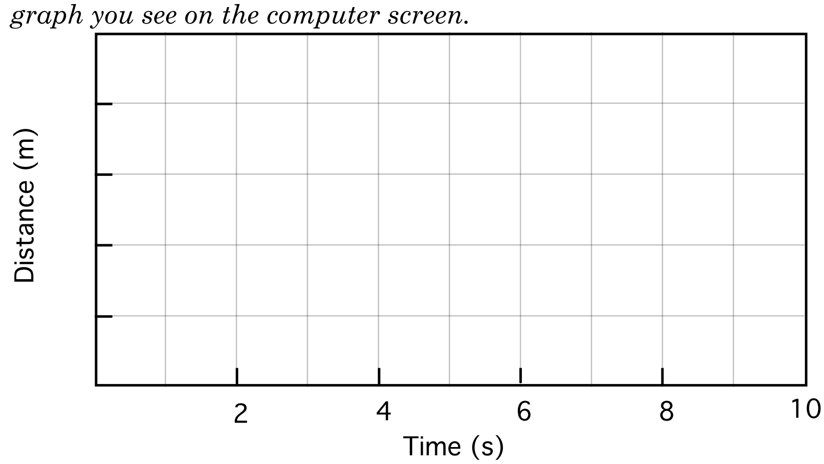
(a) What would a graph of the mass' distance from the motion detector vs. time look like? Sketch your prediction below.



(b) Explain the physical basis for your prediction.

(c) Set up the motion software to record a distance vs. time graph for 10 seconds. Take at least 100 points/second. Use the motion detector to measure the equilibrium position of the mass and record it below.

(d) Give your mass approximately the same amplitude you gave it for your casual observations. *In the space below sketch the graph you see on the computer screen.*



(e) How does the *shape* of the graph compare with that which you predicted? How about the amplitude you predicted? The period? If the observed shape differs from the predicted shape, explain what assumptions you were making that don't seem valid?

 Supported by FIPSE (U.S. Dept. of Ed.) and NSF. Modified for SFU by N. Alberding, 2005.

(f) Label the sketch of your observed graph in part (d) as follows:

"1" at the beginning of a cycle and "2" at the end the same cycle
"A" on the points on the graph where the mass is moving *away* from the detector most rapidly.
"B" on the points on the graph where the mass is moving *toward* the detector most rapidly.
"C" on the points on the graph were the mass is standing still.
"D" where the mass is farthest from the motion detector.
"E" where the mass is closest to the motion detector.

(g) Use the analysis feature of the software to read points on the graph and find the *period*, T, of the oscillations.

(h)Find the *frequency* of the oscillations, f, from the graph.

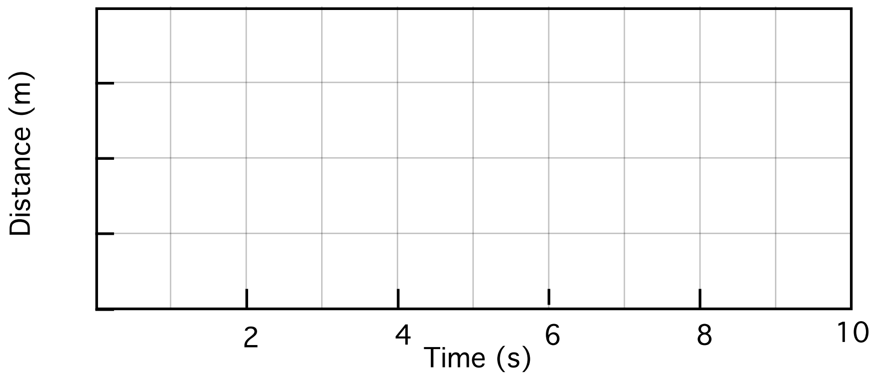
(i) Use the analysis feature again to find the equilibrium distance and the *amplitude* , X_{max}, of the oscillations. Show your computations.

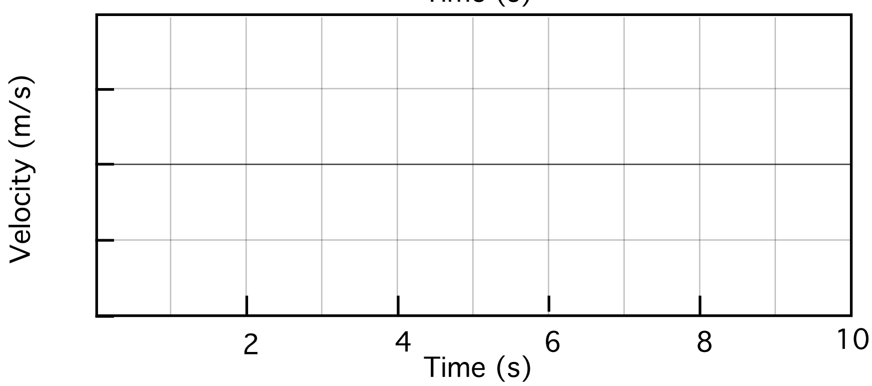
Note: At this point, be sure to save your data as a file. You will need it for the next couple of activities!

✍ Activity 14-4: Velocity Graph for a Mass-Spring System

(a) At what displacements from equilibrium is the velocity of the oscillating mass a maximum? A minimum? At what displacements is the velocity of the mass zero? *(For instance, is the velocity a maximum when the displacement is a maximum? is zero? or what?)*

(b) Use the results of your observations in part (a) to sketch a predicted shape of the graph describing how the velocity, v, of the mass varies with time compared to the variation of the displacement at the *same times*. Use a dotted line or a different colour of pen or pencil for the predicted velocity graph.





(c) Arrange to display the distance and velocity graphs on the same screen using the motion software. Sketch the actual velocity on the graph above with a solid line.

What is Simple Harmonic Motion?

Simple Harmonic Motion is defined as any periodic motion in which the displacement varies *sinusoidally* in time. In other words, either a sine or cosine function, which both have *exactly* the same basic shape, can be used to describe the displacement as a function of time. To be more exact, a general sinusoidal equation that describes the displacement x(t) as time goes on can be given in the form

$$x(t) = X_{max} \cos(\omega t + \phi) \qquad \text{[Eq. 14-1]}$$

where *A* is the *amplitude* or (maximum displacement from equilibrium) of an oscillating mass, ω is its *angular frequency*, and ϕ is its *phase angle*.

Definitions

***Angular frequency (*ω):**
$\omega \equiv 2\pi f$ (rad/s)
where ω has units of rad/s (or s^{-1}) and *f* is the frequency of oscillation in hertz

***Phase Ang*le (ϕ):**
$\phi \equiv \pm$ ArcCosine $(x(0)/X_{max})$. The phase angle is the angle in radians needed to determine the value of the displacement, $x(0)$ of the oscillating system when t=0 s. This phase angle is positive (+) when the velocity of the mass is negative and it is negative when the velocity of the mass is positive.

Was the Motion Harmonic?

During the remainder of Session 1 and continuing into Session 2, we would like you to demonstrate that, within the limits of experimental uncertainty, the actual motion of a mass on the end of a spring undergoes a sinusoidal oscillation, which can be represented by the Simple Harmonic Motion Equation (Eq. 14-1).

Consider the data you recorded earlier with the motion detector for displacement of a mass on a spring as a function of time. How closely can the data be represented by a cosine function? *This is the acid test for ideal simple harmonic motion.*

✍ Activity 14-5: Displacement vs. Time – Experimental

(a) Refer to the data you reported in Activity 14-3 and use it to find the amplitude X_{max} and the angular frequency ω associated with the motion you recorded for the mass on the spring.

X_{max} = ω =

(b) Use equation 14-1 to show mathematically that the phase angle is given by $\phi \equiv \pm$ ArcCosine $(x(0)/X_{max})$. Explain why the ± sign is needed.

(c) Use the definition of the phase angle along with the value of the displacement at time t=0 and the values of ω and A you determined to calculate the value of the phase angle, ϕ.

(d) In order to compare the theoretical and experimental displacements with each other, you should do the following:

1. Call up your file of spring oscillation data you saved earlier in this session, and paste the table of values for the distance from the motion detector vs. time for at least *two complete cycles of oscillation* into a modelling spreadsheet file starting at time t=0 seconds.
2. Add a column to the spreadsheet entitled "x(t) experimental" and calculate the displacements of the mass *from equilibrium* from the distance data. (See Figure 14-2 for details.)
3. Enter the values of X_{max}, ω and ϕ in cells.
4. Next add a column entitled "x(t) theoretical" to your spread sheet, and enter equation 14-1 to calculate the theoretical values of the displacements, x(t), at the same times you measured the experimental values of distances from the motion detector. Use absolute references to call on the cells containing values of X_{max}, ω and ϕ.
5. Format the spread sheet to include units and the correct number of significant figures for both measured and calculated values of time and displacement.

How well do the theoretical and experimental values compare?

(e) Graph both the experimental and theoretical displacements vs. time on the same graph. How do the results look? Discuss and try to explain reasons for any differences you see between the two graphs. If the agreement is less than convincing adjust the parameters of the model until the fit is acceptable.

After you get an acceptable fit upload the spreadsheet with the graphs to WebCT.

(f) On the basis of your graphs, do you think your spring-mass system underwent simple harmonic motion? Why or why not?

SESSION TWO: SIMPLE HARMONIC MOTION FOR THE MASS & SPRING

Theoretical Confirmation of SHM for a Spring-Mass System

You should be able to show that a sinusoidal motion will occur for an oscillating mass-spring system if:

(1) the one dimensional force exerted by the spring on a mass has the form $F = -kx$ where k is the spring constant, and

(2) Newton's second law holds.

Using these assumptions you can show mathematically that the following equation will hold:

$$x(\mathrm{t}) = X_{max} \cos(\omega t + \varphi)$$

where
ϕ = the phase angle indicating the displacement, $x(0)$, at $t = 0$ s
$x(t)$ = displacement of the spring from equilibrium at time t
X_{max} = amplitude of the system ,i.e., its maximum displacement

ω = the angular frequency of oscillation given by $\omega = \sqrt{\frac{k}{m}}$

where k is the spring constant and m is the oscillating mass

✍ Activity 14-6: Displacement vs. Time – Theoretical

(a) Show that the equation of motion ($F = ma$ when written out) of a mass m attached to a spring of force constant k is given by

$$m\frac{d^2x}{dt^2} = -kx$$

Hint: (1) What are the definitions of instantaneous velocity and acceleration in one dimension? (2) $x(t)$ is really just x expressed in a form that reminds us that x changes with t.

Note: This type of equation, which occurs frequently in physics, is known as a *differential equation*.

(b) Show that if

$$d\frac{\cos(\omega t+\phi)}{dt} = -\omega\sin(\omega t+\phi)$$

and

$$d\frac{\sin(\omega t+\phi)}{dt} = \omega\cos(\omega t+\phi)$$

where $\omega^2 = k/m$

then the equation $x(\mathrm{t}) = X_{max}\cos(\omega t+\phi)$

satisfies the equation $m\frac{d^2x}{dt^2} = -kx$

A Mathematical Model for the Mass-Spring System.

You have confirmed the fact that the theoretical equation describing a mass-spring system is given by

$$x(t) = X_{max}\cos(\omega t+\phi)$$

where

ϕ = the phase angle indicating the displacement, $x(0)$ at $t = 0$ s
$x(t)$ = displacement of the spring from equilibrium
X_{max} = amplitude of the system i.e., its maximum displacement

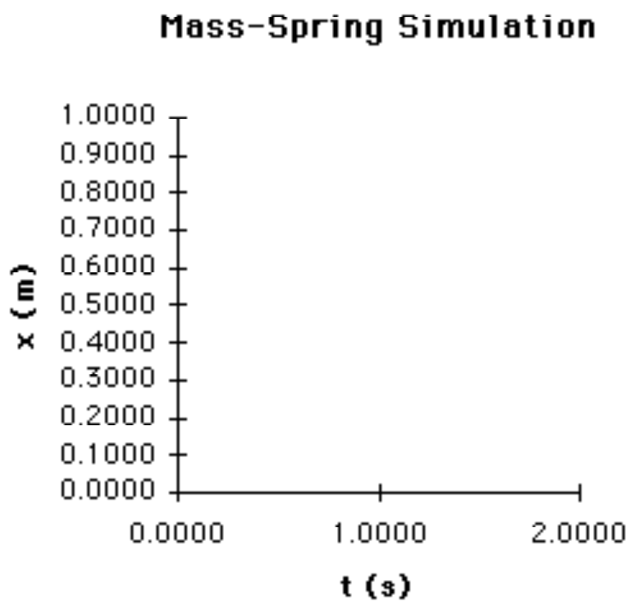
ω = the angular frequency of oscillation given by $\omega = \sqrt{\frac{k}{m}}$

where k is the spring constant and m is the oscillating mass

In this activity you will construct a spreadsheet model to explore the behaviour of mass-spring systems for different values of the four parameters X_{max}, ϕ, k, and m. You can start by using the Excel Modelling Worksheet and setting appropriate column headings and parameter labels as shown in the illustration below.

Mass-Spring Simulation

	A	B	C
1		Xmax(m)=	1.0000
2		k (N/m) =	10.0000
3		m (kg) =	0.2000
4		ø (rad) =	1.0000
5	**Mass-Spring Simulation**		
6	x (m)		
7	t (s)	x(m) theo	x(m) exp
8	0.0000		
9	0.0500		
10	0.1000		
11	0.1500		
12	0.2000		
13	0.2500		
14	0.3000		
15	0.3500		
16	0.4000		
17	0.4500		
18	0.5000		
19	0.5500		
20	0.6000		
21	0.6500		



To construct your mathematical model you should do the following:

1. Open the modelling worksheet and label the parameters, column headers, etc. as shown above.

2. Get set to do calculations every 1/20th of a second for two seconds by entering times of 0.000 s, 0.0500 s, etc. in the time column.

3. Enter the equation of motion $x(t) = X\cos\left(\sqrt{\frac{k}{m}}\,t + \phi\right)$

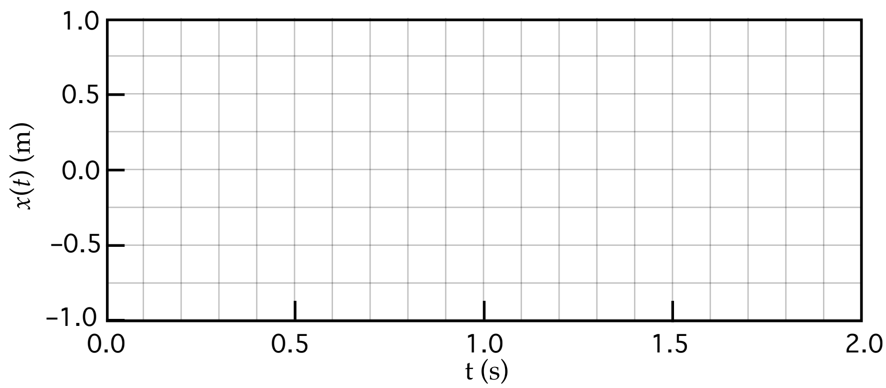
in the x-th (x theoretical column). Get the values of k, m, etc. from cells C1, C2, C3, and C4. Don't forget to call on these cells with absolute references i.e. C1, C2, etc.

4. Place reasonable values for the four parameters in the appropriate cells. The values in the sample table will get you started. Once this is done you should see a cosine function on the graph.

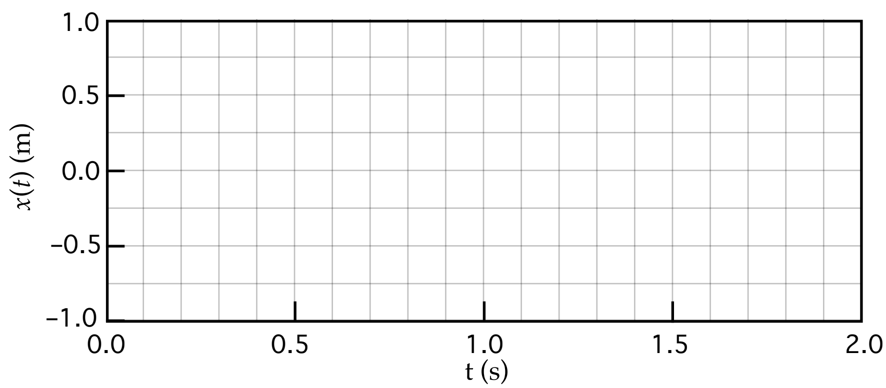
✍ Activity 14-7: The Spreadsheet Model–Outcomes

(a) Use your simulation to graph for X_{max} = 1.0000 m, k= 10.00000 N/m, m=0.2000 m, and ϕ =1.0000 rad in the space below. Sketch the graph in the space below.

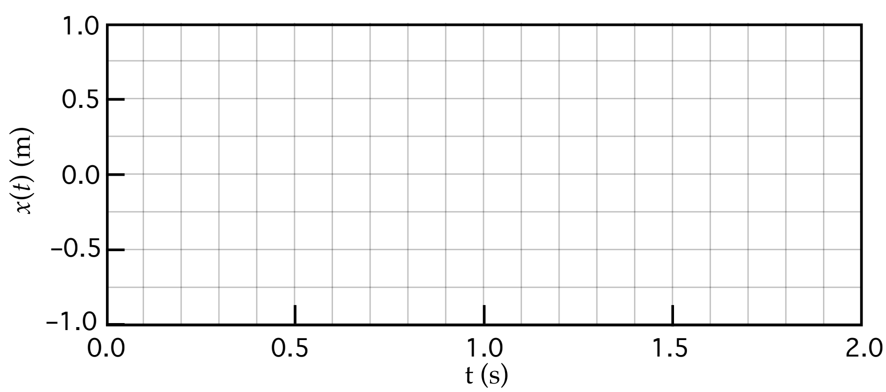
(b) What do you predict will happen to the graph if X_{max} is decreased to 0.5000 m? Sketch your prediction on the graph below, using a dotted line. Explain the reasons for your prediction. Now, try decreasing it in your simulation and sketch the result on the same graph, using a solid line. **Beware**: Do your sketches on the same scale – the Excel program might rescale your graph automatically!



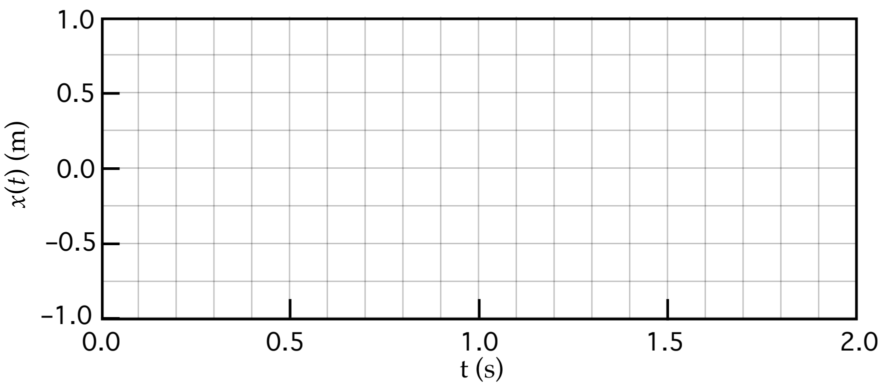
(c) Reset X_{max} to 1.0000 m. What do you predict will happen to the graph if k is decreased to 5.0000 N/m? Again using a dotted line, sketch your prediction on the graph below. Explain the reasons for your prediction. Now try decreasing it in your simulation and sketch the result using a solid line. **Beware**: Do your sketches on the same scale – the Excel program might rescale your graph automatically.



(d) Reset k to 10.000 N/m. What do you predict will happen to the graph if m is decreased to .1000 kg? Sketch your prediction below. Explain the reasons for your prediction. Try decreasing it in the simulation and sketch the result. **Beware**: Do your sketches on the same scale – the Excel program might rescale your graph automatically.



(e) Reset m to its original value. What do you predict will happen to the graph if ϕ is decreased to 0.5000 rad? Sketch your prediction below, and explain the reasons for your prediction. Try decreasing it in your simulation and sketch the result. **Beware**: Do your sketches on the same scale – the Excel program might rescale your graph automatically.



(f) Find the value for the spring constant k from the value of ω and m you found for the spring-mass system of session one.

SESSION THREE: THE SIMPLE PENDULUM

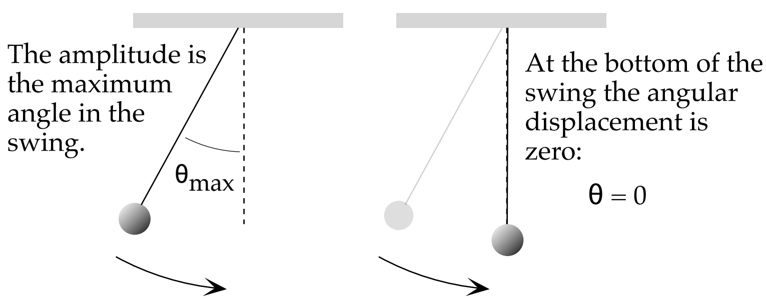
What Does the Period of a Pendulum Depend On?

When a mass suspended from a string is raised and released it oscillates. The oscillating motion of a simple pendulum has been used throughout history to record the passage of time. How do clock makers go about constructing a pendulum with a given period? Why does a pendulum oscillate? What factors affect its period? For the following activity, you will need:

- Pendulum parts including:
  - bobs (small round objects with different masses)-
  - string-
  - stand to suspend the pendulum
- Timing devices: (your choice, if available)
  - a stop watch
  - a photogate timing system
  - a motion detection system
  - a video analysis system
  - a rotary motion detector

✍ Activity 14-8: Factors Influencing Pendulum Period

(a) Watch the oscillation of a pendulum carefully. Sketch the forces on the bob when the pendulum is at its maximum angular displacement and at zero displacement.



.

(b) Explain why the pendulum oscillates back and forth when the bob is lifted through an angle θ_{max} and released.

(c) List *all* the factors that might conceivably affect the period of a pendulum in the table below. Indicate for each factor whether or not you expect it to matter. Feel free to discuss your ideas with your classmates and debate the issues.

Factor	increase	decrease	none	Reasons for your prediction	Correct?
	☐	☐	☐		
	☐	☐	☐		
	☐	☐	☐		
	☐	☐	☐		

(d) Play with the pendulum factors and see which factors obviously matter. Summarize your findings in the space below and indicate in the table above which of your predictions were correct.

(e) [7 pts] Design an experiment to check the possible dependence of the period of a pendulum on its length in a much more careful quantitative way. You have a choice of a number of timing devices to complete this task. Regardless of which device you use you should design your experiment to be as accurate as possible with your time measurements. Document your experiment on the 8-page mini-lab notebook that we provide. **Hint:** Be sure to use some very short lengths and some much longer lengths for your pendulum.

The lab write up should contain the following

1. Title
   - Experiment Name
   - Date
   - Your name and your partners'
2. Experiment's objective
3. Description of the apparatus and procedure. A simple diagram that is labelled is needed here.
4. Experimental results.
   - Put data in a neat table. Include uncertainty estimates.
   - Write the results Directly into the table as you take the data. Avoid recopying data from scrap paper.
   - Do not erase data. If you need make a correction, cross out the data and rewrite the correct values.
5. Calculations and results.
   - Display results graphically if possible.
   - Show sample calculations.
6. Conclusions: How do your results compare with your prediction?

Should the Pendulum Really Undergo SHM?

In your theoretical consideration of the mass-spring system, you showed mathematically that, if the restoring force is proportional to the displacement but opposite in direction, then one would expect to see the mass undergo simple harmonic motion. The restoring force for the spring has the form $F=-kx$. To what extent does the restoring force for a simple pendulum which oscillates at a small angle of displacement have a similar mathematical form to that of the mass on a spring? In this next activity you will derive the equation of motion for a simple pendulum. This equation is very similar to the equation of motion of the mass-spring system, and so it will be clear that the simple pendulum ought to undergo a simple harmonic motion in which its period of motion is independent of the mass of the pendulum bob.

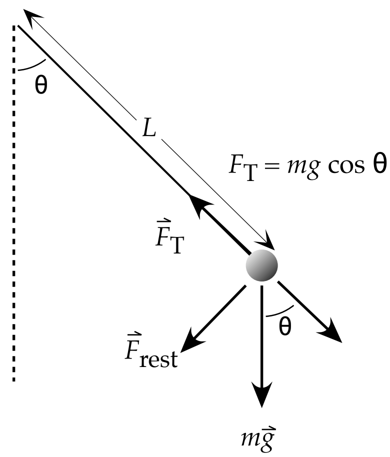
In order to derive the equation of motion you should recall the when a mass, *m*, experiences a torque, it will undergo an angular acceleration given by the equation

$$\tau = I\alpha$$

You can look at the form of this equation for a simple pendulum when the angle of the oscillation is small. You should find that it is quite similar to the equation for the mass-spring system.

✍Activity 14-9: The Pendulum Equation of Motion

(a)What is the restoring force on the pendulum bob as a function of m, g, and the displacement angle θ?



(b) *The small angle approximation:* The value of θ in radians and the value of sinθ are quite close to each other for small values of θ. Use a spreadsheet or scientific calculator to find the angle in radians for which θ and sin θ vary from each other by 1%. Use three decimal places in your calculations.

(c) Calculate the value in degrees of the angle you have just calculated in radians.

(d) Show that, if the maximum angle through which the pendulum swings is small enough so that $\theta \approx \sin\theta$ (say to within about 1%), then the restoring force can be expressed (to within 1%) by $F = -mg\theta$.

(c) Show that the torque experienced by the mass is given by the expression $\tau = -mg\theta L$ (where L is the length).

(d) What is the rotational inertia of the simple pendulum as a function of its mass, m, and length, L?

(e) Use the relationship between τ, I, and α to show that the equation of motion for the angular displacement of the pendulum is given by [you fill in the blank]

(g) How does this compare to the equation of motion you derived

for the mass-spring system given by $m\frac{d^2x}{dt^2} = -kx$

What is the same? What is different?

(h) Refer to the solution of the spring-mass equation of motion to write down the solution to the pendulum equation and show why its solution is given by

Hint: In the pendulum equation of motion the term θ plays the role of x in the mass-spring equation and the term (mg/L) plays the role of the spring constant k.

(i) Show that if the period, T, of a mass-spring system is given by the equation

Spring-Mass: $T = 2\pi\sqrt{\frac{m}{k}}$

then the period of the pendulum ought to be given by

Pendulum: $T = 2\pi\sqrt{\frac{L}{g}}$

(j) How does this expression for the period compare to the one which you found experimentally in Activity 14-7 (e)?

(k) Many people are surprised to find that the period of a simple pendulum does not depend on its mass. Can you explain why the period of a simple pendulum doesn't depend on its mass? **Hint:** Can you explain why the acceleration of a falling mass close to the surface of the earth is a constant regardless of the size of the mass?