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2. Overview

12-2 Combining Translations with Simple Rotations

- A **rolling** object, such as a yo-yo or a wheel, rotates as its center of mass moves along a line. The center of mass obeys Newton's second law: $\vec{F}^{\text{net}} = M\vec{a}_{\text{com}}$, where $\vec{F}_{\text{net}}$ is the net force on the object, M is the mass of the object, and $\vec{a}_{\text{com}}$ is the acceleration of the center of mass. Rotation about an axis through the center of mass is governed by Newton's second law for rotation: $\tau^{\text{net}} = I\alpha$, where τ^{net} is the net torque on the object, I is the rotational inertia of the object, and α is its rotational acceleration.
- The two motions are related by the forces that act on the object. The net force accelerates the center of mass and the net torque (derived from the forces) produces a rotational acceleration. If, in addition to the mass and rotational inertia of the body, you know all the forces acting and their points of application you can calculate the acceleration of the center of mass and the rotational acceleration about the center of mass.
- The velocity of the point on a rolling wheel in contact with the surface is given by the vector sum of the velocity due to the motion of the center of mass and the velocity due to rotation. If the wheel is rolling along the x axis $v_x = v_{\text{com}\ x} - \omega R$, where R is the radius of the wheel and the forward direction was taken to be positive. If the wheel slides, then $v_x \neq 0$ and $v_{\text{com}\ x}$ is different from ωR. If it does not slide, then $v_x = 0$ and $v_{\text{com}\ x} = \omega R$. Furthermore, if the wheel accelerates without sliding, the magnitude of the acceleration of the center of mass a_{com} is related to the magnitude of the rotational acceleration around the center of mass by $a_{\text{com}} = \alpha R$.
- For a yo-yo $a_{\text{com}} = \alpha R_0$, where R_0 is the radius of the axle.
- The kinetic energy of an object that is simultaneously rotating and translating is the sum of its translational and rotational kinetic energies: $K = \frac{1}{2}mv_{\text{com}}^2 + \frac{1}{2}I\omega^2$.

12-3 Rotational Variables as Vectors

- Rotational velocity and rotational acceleration, but not rotational position or displacement, can be considered to be vectors along the axis of rotation. To determine the direction of $\vec{\omega}$, use the right hand rule: curl the fingers of your right hand around the axis in the direction of rotation. Your thumb then points in the direction of $\vec{\omega}$.
- If the body rotates faster as time goes on, then $\vec{\alpha}$ and $\vec{\omega}$ are in the same direction. If it rotates more slowly, then $\vec{\alpha}$ and $\vec{\omega}$ are in opposite directions.
- Although directions can be assigned arbitrarily to rotational displacements, they do not add as vectors. The result of two successive rotational displacements around different axes that are not parallel, for example, depends on the order in which the rotational displacements are carried out. Thus, the rules of vector addition are not obeyed.
- Torque is also a vector.

12-4 The Vector or Cross Product

- The **vector product** (or cross product) of two vectors, written $\vec{a} \times \vec{b}$, is a vector. Its magnitude is given by $|\vec{a} \times \vec{b}| = ab\sin\phi$, where ϕ is the angle between $\vec{a}$ and $\vec{b}$ when they are drawn with their tails at the same point. ϕ is always in the range from 0 to $180°$ and the magnitude of the vector product is always positive.
- The direction of the vector product is always perpendicular to both of the factors. To find the direction of $\vec{a} \times \vec{b}$, draw the vectors with their tails at the same point and pretend there is a hinge at that point. Curl the fingers of your right hand so they rotate $\vec{a}$ into $\vec{b}$. Your thumb will then point in the direction of $\vec{a} \times \vec{b}$. Note that $\vec{b} \times \vec{a} = -\vec{a} \times \vec{b}$.
- In terms of components $\vec{a} \times \vec{b} = (a_y b_z - a_z b_y)\,\hat{i} + (a_z bx - a_x b_z)\,\hat{j} + (a_x b_y - a_y b_z)\,\hat{k}$.
- The vector product of two vectors with given magnitudes is zero if the vectors are parallel or antiparallel ($\phi = 0$ or $180°$); it has its maximum value if they are perpendicular to each other ($\phi = 90°$).

12-5 Torque as a Vector Product

- When a force $\vec{F}$ is applied to an object at a point with position vector $\vec{r}$, relative to some origin, then the **torque** $\vec{\tau}$ associated with the force is given by the vector product $\vec{\tau} = \vec{r} \times \vec{F}$. The magnitude of the torque is given by $\tau = rF \sin\phi$, where ϕ is the angle between $\vec{F}$ and $\vec{r}$ when they are drawn with their tails at the same point. A right-hand rule gives the direction of the torque: curl the fingers of the right hand so they rotate $\vec{r}$ into $\vec{F}$ about the point where the tails meet, through the smallest angle between them. The thumb then points in the direction of the torque.
- The torque acting on an object is different for different choices of the origin used to describe the position $\vec{r}$ of the application point.
- For an object rotating around a fixed axis, only the component of the torque along the axis is important. For a force that lies in a plane perpendicular to the axis, this component is given by $RF_\perp$, where R is the distance from the axis of rotation to the point of application of the force and $F_\perp$ is the tangential component of the force. That is, it is the component along a line that is tangent to the circular path of the application point.

12-6 Rotational Form of Newton's Second Law

- The rotational momentum of a particle with position vector $\vec{r}$ and translational momentum $\vec{p}$ is defined by the vector product $\vec{\ell} = \vec{r} \times \vec{p}$.
- If a net torque acts on a particle, then its rotational momentum changes with time. Recall that, in terms of the momentum of a particle, Newton's second law can be written $\vec{F}_{\text{net}} = d\vec{p}/dt$, where $\vec{F}_{\text{net}}$ is the net force on the particle. Take the vector product of this equation with the position vector $\vec{r}$ of the particle to obtain $\vec{r} \times \vec{F}_{\text{net}} = \vec{r} \times d\vec{p}/dt$. Note that $\vec{r} \times \vec{F}_{\text{net}} = \vec{\tau}_{\text{net}}$, the net torque acting on the particle. With a small amount of mathematical manipulation you can show that $\vec{r} \times d\vec{p}/dt = d\vec{\ell}/dt$. Thus, in terms of torque and rotational momentum, Newton's second law becomes

$$\vec{\tau}_{\text{net}} = \frac{\vec{\ell}}{dt} .$$

12-7 Rotational Momentum

- The rotational momentum of a particle was defined by $\vec{\ell} = \vec{r} \times \vec{p}$, where $\vec{r}$ is the position vector of the particle and $\vec{r}$ is its momentum. The magnitude is $\ell = rp \sin\phi$, where ϕ is the angle between $\vec{r}$ and $\vec{p}$ when they are drawn with their tails at the same point. The direction is given by the right hand rule for vector products.
- If $p_\perp$ is the component of $\vec{p}$ along an axis that is in the plane of $\vec{r}$ and $\vec{p}$ and is perpendicular to $\vec{r}$, then $\ell = rp_\perp$.
- If $r_\perp$ is the component of $\vec{r}$ along an axis that is in the plane of $\vec{r}$ and $\vec{p}$ and is perpendicular to $\vec{p}$, then $\ell = r_\perp p$.
- A particle moving along a straight line that does not pass through the origin has a non-zero rotational momentum about the origin. Its magnitude is given by $\ell = mvd$, where m is the mass of the particle, v is its speed, and d is the distance from the origin to the line of motion.
- If a particle of mass m is traveling around a circle of radius R with speed v and the origin is placed at the center of the circle, then the magnitude of its rotational momentum is $\ell = mRv$. Curl the fingers of the right hand around the axis in the direction the particle is traveling. Then, the thumb points along the axis in the direction of the rotational momentum.

12-8 The Rotational Momentum of a System of particles

- The total rotational momentum $\vec{L}$ of a system of particles is the vector sum of the individual rotational momenta of all the particles in the system: $\vec{L} = \sum \vec{\ell}_i$, where $\vec{\ell}_i$ is the rotational momentum of particle i.
- For a system of particles, Newton's second law for rotation becomes

$$\vec{\tau}_{\text{net}} = \frac{d\vec{L}}{dt} ,$$

where $\vec{\tau}_{net}$ is the net external torque on particles of the system. This is the fundamental equation for a system of particles. Notice that only an external torque can change the total rotational momentum of a system. Torques exerted by particles of the system on each other might change the individual rotational momenta but not the total.

12-9 The Rotational Momentum of a Rigid Body Rotating About a Fixed Axis

- For an object rotating with rotational velocity $\vec{\omega}$ about the z axis, the z component of the rotational momentum, in terms of ω_z, is the sum over all particles in the object $L_z = \sum m_i r_{\perp i} v_i = \sum m_i r_{\perp i}^2 \omega_z$, where m_i is the mass of particle i, and $r_{\perp i}$ is its distance from the rotation axis. In terms of the rotational inertia I of the object, it is $L_z = I\omega_z$.

12-10 Conservation of Rotational Momentum

- If the net external torque on a body is zero, then the change over any time interval of its rotational momentum is zero. Rotational momentum is then said to be conserved. This is a vector law. One component of the rotational momentum may be conserved while other components are not. When attempting to use the conservation principle, you must first check to be sure the net external torque or one of its components vanishes.
- If the body rotates about the z axis and the net external torque is zero then $I\omega_z$ is constant. Consider an ice skater, initially rotating on the points of her skates with her arms extended. By dropping her arms to her sides she decreases her rotational inertia and her rotational velocity must increase for her rotational momentum to remain the same.
- If the rotational inertia of the body changes, write $I_1\vec{\omega}_1$ for the rotational momentum in terms of the initial rotational inertia and rotational velocity and $I_2\vec{\omega}_2$ for the rotational momentum in terms of the final rotational inertia and rotational velocity. If no external torques act, these are equal. Solve $I_1\vec{\omega}_1 = I_2\vec{\omega}_2$ for an unknown.
- If two objects exert torques on each other and no external torques act, write the total rotational momentum in terms of the initial rotational velocities ($I_A\vec{\omega}_{A1} + I_B\vec{\omega}_{B2}$), then write it again in terms of the final rotational velocities ($I_A\vec{\omega}_{A2} + I_B\vec{\omega}_{B2}$). Equate the two expressions and solve for an unknown.
- In some problems, one object is initially moving along a straight line, as when a child runs and jumps on a merry-go-round. Don't forget to include the rotational momentum of this object.