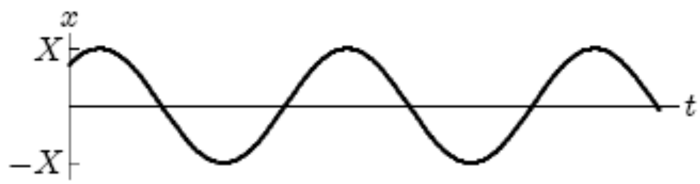
Print this page

2. Overview

16-2 The Mathematics of Sinusoidal Oscillations

- If an object moves along the x axis in **simple harmonic motion**, its coordinate x as a function of time t is given by



$$x(t) = X\cos(\omega t + \phi_0)\ ,$$

where X, ω, and ϕ_0 are constants. A possible function $x(t)$ is graphed above.

- The object moves back and forth between $x = -X$ and $x = +X$. X is called the **amplitude** of the oscillation.
- The constant ω is called the **angular frequency** of the oscillation. Since ωt is measured in radians, the unit of ω is rad/s. The **frequency** f of the oscillation gives the number of times the object moves through a complete cycle per unit time. It is measured in hertz ($1\ Hz = 1\ s^{-1}$). The frequency and angular frequency are related by $\omega = 2\pi f$. The **period** T of the oscillation is the time for one complete cycle. It is related to the angular frequency and frequency by $T = 1/f$ and $T = 2\pi/\omega$.
- The combination $\omega t + \phi$ is called the **time-dependent phase** or just plain **phase** of the oscillation and ϕ_0 is called the **phase constant** or **phase angle**. It is the value of the phase for $t = 0$. A change in the phase constant simply moves the curve shown on the graph above left or right along the t axis. The phase and phase constant are usually measured in radians.

16-3 Simple Harmonic Motion: The Mass-Spring System

- If a block and attached to the end of an ideal spring and moves along the x axis then the x component of the spring force on the block is $F_x^{\text{spring}} = -kx$, where k is the spring constant. The origin was placed at the position of the block when the spring is neither extended or compressed.
- If the spring force is the only force on the block Newton's second law gives $-kx = ma_x$, where a_x is the x component of the block's acceleration and m is the mass of the block. The solution to $d^2x/dt^2 = -kx$ is $x(t) = X\cos(\omega t + \phi)$, where $\omega = \sqrt{k/m}$ and X and ϕ are constants. The angular frequency of a block on a spring is determined by the spring constant of the spring and the mass of the block.
- The period of oscillation of the block is given by $T = 2\pi\sqrt{m/k}$.
- Newton's second law tells us that the force that must be applied to an object of mass m to produce simple harmonic motion is $F(t) = ma(t) = -m\omega^2 x(t)$. The force must be proportional to the displacement from equilibrium and the constant of proportionality must be negative.
- Consider a block on the end of spring, moving on a frictionless horizontal surface. If the origin is taken to be at the position of the block when the spring is neither extended nor compressed, the force of the spring on the block is given by $F = -kx$, where k is the spring constant. Since the force is proportional to the displacement and the constant of proportionality is negative we know the motion is simple harmonic. Furthermore, a comparison of $F = -kx$ with $F = -m\omega^2 x$ tells us that the angular frequency of the motion is $\omega = \sqrt{k/m}$. The period is $T = 2\pi/\omega = 2\pi\sqrt{m/k}$.
- A rotational simple harmonic oscillator consists of an object suspended by a wire that exerts a torque when it is twisted. The torque is proportional to the angle of twist and the constant of proportionality is negative; it is a restoring torque. If the angular position θ of the object is measured from the its position when the wire is not

twisted, then $\tau = -\kappa\theta$, where κ is called the torsion constant of the wire. Newton's second law for rotation becomes $-\kappa\theta = I\alpha$, where I is the rotational inertia of the object. This equation is exactly like the equation for a mass on a spring, except that θ has replaced x, α has replaced a, I has replaced m, and κ has replaced k. The object rotates back and forth in simple harmonic motion with angular frequency $\omega = \sqrt{\kappa/I}$ and period $T = 2\pi/\omega = 2\pi\sqrt{I/\kappa}$.

16-4 Velocity and Acceleration for SHM

- An expression for the velocity of the object as a function of time can be found by differentiating the expression for $x(t)$ with respect to time:

  $$v(t) = \frac{dx(t)}{dt} = -\omega X \sin(\omega t + \phi_0)\ .$$

  The maximum speed of the object, called the velocity amplitude, is given by $V = \omega X$. The object has maximum speed when its coordinate is $x = 0$. The velocity of the object is zero when its coordinate is $x = -X$ and also when its coordinate is $x = +X$.

- An expression for the acceleration of the object as a function of time can be found by differentiating $v(t)$ with respect to time:

  $$a(t) = \frac{dv(t)}{dt} = -\omega^2 X \cos(\omega t + \phi) = -\omega^2 x(t)\ .$$

  The magnitude of the maximum acceleration, call the acceleration amplitude, is $A = \omega^2 X$. The object has maximum acceleration when its coordinate is $x = -X$ and also when its coordinate is $x = +X$. This is where the velocity vanishes. The acceleration is zero when the coordinate is $x = 0$. This is where the speed is a maximum.

- The amplitude X and phase constant ϕ are determined by the initial conditions (at $t = 0$). Since $x(t) = x\cos(\omega t + \phi)$ the initial coordinate is given by $x_0 = X\cos\phi$ and the initial velocity is given by $v_0 = -\omega X \sin\phi$.

  x_0 is positive for $-\pi/2\,rad < \phi < \pi/2\,rad$.

  x_0 is negative for $\pi/2\,rad < \phi < 3\pi/2\,rad$.

  v_0 is positive for $\pi\,rad < \phi < 2\pi rad$.

  v_0 is negative for $0 < \phi < \pi\,rad$.

- The equations $x_0 = X\cos\phi$ and $v_0 = -\omega X\sin\phi$ can be solved for X and ϕ. To obtain an expression for X, solve the first equation for $\cos\phi$ and the second for $\sin\phi$, then use the trigonometric identity $\cos^2\phi + \sin^2\phi = 1$. To obtain an expression for ϕ, divide the second equation by the first and solve for $\tan\phi$. The results are

  $$X = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}} \qquad and \qquad tan\phi = -\frac{v_0}{\omega x_0}$$

- Be careful when you evaluate the expression for ϕ. There are always two angles that are the inverse tangent of any quantity, but your calculator only gives the one closest to 0. The other is $180°$ or π radians away. Always check to be sure $X\cos\phi$ gives the correct initial coordinate and $-\omega X\sin\phi$ gives the correct initial velocity. If they do not, add $\pi\,rad$ to the value you used for ϕ.

16-5 Gravitational Pendula

- A **simple pendulum** consists of a small object of mass m suspended by a string. After the mass is pulled aside and released, it swings back and forth along the arc of a circle with radius equal to the length L of the string. If the z axis is perpendicular to the plane of the swing then, when the string makes the angle θ with the vertical, the z component of the gravitational torque about the pivot point is $\tau_z = -mgL\sin\theta$, where the negative sign indicates that the torque is pulling the mass toward the $\theta = 0$ position. Newton's second law for rotation becomes $-mgL\sin\theta = I\alpha$, where I is the rotational inertia and α is the rotational acceleration. The rotational acceleration is not proportional to the rotational displacement and the motion is not strictly simple harmonic. However, if θ in radians is small, then $\sin\theta$ can be approximated by θ itself and the equation becomes $-mgL\theta = I\alpha$. If θ is always small, the motion is very nearly simple harmonic, the angular frequency is $\omega = \sqrt{mgL/I}$, and the period is $T = 2\pi/\omega = 2\pi\sqrt{I/mgL}$. For a simple pendulum $I = mL^2$, so $\omega = \sqrt{g/L}$ and $T = 2\pi\sqrt{L/g}$.

- A **physical pendulum** consists of an object that is pivoted about some point other than its center of mass. The analysis is like that carried out for a simple pendulum. If h is the distance from the pivot point to the center of mass, the torque acting on the object is $\tau = -mgh\sin\theta$, where θ is the angle between the vertical and the line that joins the pivot and center of mass. If θ is small, $\sin\theta$ can be approximated by θ itself, in radians. Then $-mgh\theta = I\alpha$. The angular acceleration is proportional to the angular displacement and the constant of proportionality is negative, so the motion is simple harmonic. The angular frequency is $\omega = \sqrt{mgh/I}$ and the period is $T = 2\pi\sqrt{I/mgh}$.

16-6 Energy in Simple Harmonic Motion

- For the block on the end of a spring, an expression for the spring potential energy as a function of time can be found by substituting $x(t) = X\cos(\omega t + \phi)$ into $U = \frac{1}{2}kx^2$:
$$U(t) = \tfrac{1}{2}kX^2\cos^2(\omega t + \phi)\ .$$
An expression for the kinetic energy as a function of time can be found by substituting $v = -\omega X\sin(\omega t + \phi)$ into $K = \frac{1}{2}mv^2$:
$$K(t) = \tfrac{1}{2}m\omega^2X^2\sin^2(\omega t + \phi)\ .$$
If $\omega^2 = k/m$ is used, this can also be written
$$K(t) = \tfrac{1}{2}kX^2\sin^2(\omega t + \phi)\ .$$
- Both the potential and kinetic energies vary with time. The potential energy is a maximum when the coordinate is $x = -X$ and also when it is $x = +X$. Then the speed is zero and the kinetic energy vanishes. The potential energy is a minimum when $x = 0$. Then the speed is the greatest and the kinetic energy is a maximum. Notice that the maximum kinetic energy has exactly the same value as the maximum potential energy.
- Although both the potential and kinetic energies vary with time, the total mechanical energy $E^{\text{mec}} = K + U$ is constant, as you can see by adding the expressions given above and using $\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi) = 1$. All of the following expressions for the mechanical energy give the same value:
$$E^{\text{mec}} = \tfrac{1}{2}mv^2(t) + \tfrac{1}{2}kx^2(t) = \tfrac{1}{2}kX^2 = \tfrac{1}{2}mV^2\ .$$
You will solve some problems by equating two of these expressions and solving for one of the quantities in them.

16-7 Damped Simple Harmonic Motion

- Many oscillating systems in nature are damped by a force that is proportional to the velocity. Consider a mass m on the end of spring with spring constant k and subject to the damping force $-b\vec{v}$, where b is a damping constant and $\vec{v}$ is the velocity. If the motion is along the y axis, with $y = 0$ corresponding to an unstretched spring, Newton's second law becomes $-ky - bv = ma$. The solution is
$$y(t) = Ye^{-bt/2m}\cos(\omega' t + \phi)\ ,$$
where

$$\omega' = \sqrt{\left(\frac{k}{m}\right)^2 - \frac{b^2}{4m^2}}\,.$$

If $b < 4km$, then the motion is oscillatory but the amplitude ($Y\,e^{-bt/2m}$) decreases exponentially with time. See Fig. 16-26 of the text for a graph of $y(t)$.

- If $b \geq km$, then no oscillations take place. Instead, the spring settles back to its unstretched length and remains at that length.
- The total mechanical energy of a damped oscillator is not constant. As time goes on, the damping force dissipates energy.

16-8 Forced Oscillations and Resonance

- A mass on the end of spring can be driven to oscillate in simple harmonic motion by applying an external force of the form $F_m \cos\omega t$. In this section, the angular frequency of the force is denoted by ω_d and the natural angular frequency ($\sqrt{k/m}$) of the spring-mass system is denoted by ω.
- The angular frequency of a forced oscillation is the same as the angular frequency of the driving force and is NOT necessarily the natural angular frequency of the oscillator.
- The amplitude of the oscillation depends on the driving frequency, as well as on the driving amplitude F_m. The driving frequency for which the velocity amplitude is a maximum is the same as the natural frequency of the oscillator and is called the **resonance frequency**. It is nearly the same as the frequency for which the amplitude is a maximum. Fig. 16-31 of the text shows how the amplitude depends on the driving frequency.
- The amplitude of the motion does not decay with time even though a resistive force is present and energy is being dissipated. The mechanism that drives the oscillator and provides the external force does work on the system and supplies the energy required to keep it going at constant amplitude.