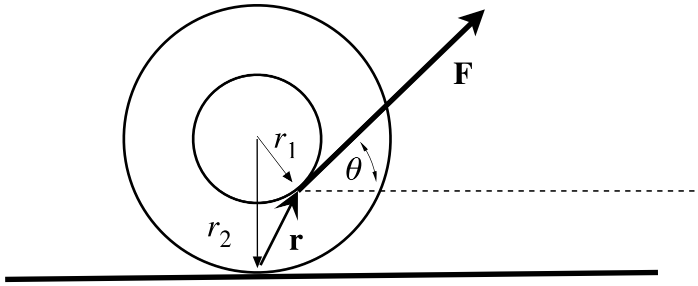
Challenge Problems:

Yo-Yo

A yo-yo is pulled by its string on the floor without slipping. Depending on the angle of the string with respect to the horizontal, the yo-yo will travel towards the puller or away from the puller. Find the angle from the horizontal the string has to be pulled which demarcates the forward movement and backward movement.



Solve for the acceleration, either angular or translational.

Solution 1:

Consider rotation to occur about a pivot at the contact point with the horizontal. Let **F** be the pulling force on the string. The torque about the that point is given by **r** x **F.**

Using some simple trig we can express the vectors for **r** and **F** in terms of rectangular coordinates.

$$\vec{r} = r_1 \sin\theta\hat{i} + (r_2 - r_1\cos\theta)\hat{j}$$
$$\vec{F} = F\cos\theta\hat{i} + F\sin\theta\hat{j}$$

Now just do the cross product to get the torque and simplify. ($\hat{k}$ would be out of the paper.)

$$\begin{aligned}\vec{\tau} &= \vec{r}\times\vec{F} = (xF_y - yF_x)\hat{k}\\ &= [r_1\sin\theta F\sin\theta - (r_2 - r_1\cos\theta)(F\cos\theta)]\hat{k}\\ &= (r_1F\sin^2\theta - r_2F\cos\theta + r_1F\cos^2\theta)\hat{k}\\ &= (r_1F - r_2F\cos\theta)\hat{k}\end{aligned}$$

Let the moment of inertia of the yo-yo be expressed in terms of the outer radius:

$$I_{\text{cm}} = cmr_2^2$$

Where *c* is just a constant less than one depending on the mass distribution of the yoyo. Using the parallel axis theorem to get the *I* about the pivot point at the contact between the surface and the rim:

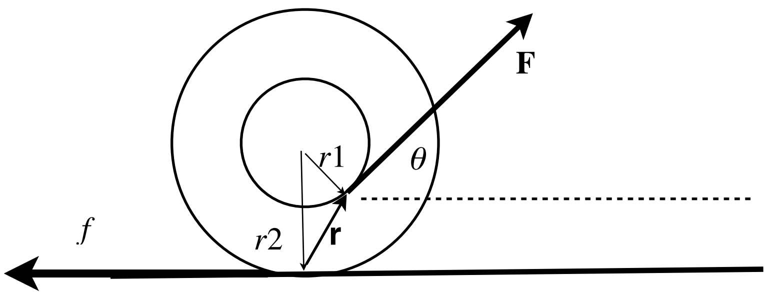
$$I = (c+1)mr_2^2$$

Thus the angular acceleration is

$$\alpha = \frac{F(r_1 - r_2\cos\theta)}{(c+1)mr_2^2}$$

Here positive α would correspond to rolling to the left, away from the puller. This occurs when r_1/r_2 is greater than cos θ.

Solution 2:



Consider the rotation about the centre of mass and let the static friction force be represented by *f*.

The net force on the object on the c.m. is

$$F_{\text{net}} = F\cos\theta - f = ma$$

The net torque about c.m. is (magnitude only)

$$\tau = Fr_1 - fr_2 = I_{\text{cm}}\alpha$$

Substituting for *f* and I we get

$$\tau = Fr_1 - (F\cos\theta - ma)r_2 = cmr_2^2(-a/r_2)$$

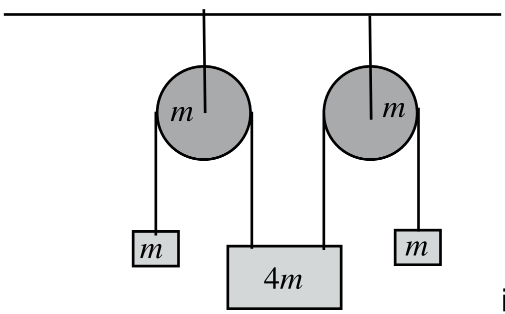
Notice that $\alpha = -a/r_2$ in order for the signs to be consistent.

Solving for *a*

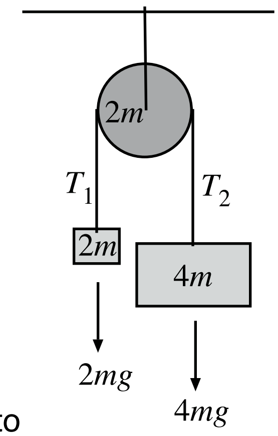
$$a = -\frac{F(r_1 - r_2\cos\theta)}{(c+1)mr_2}$$

This is consistent with solution 1. When cos θ is greater than r_1/r_2 then *a* is positive.

Double Atwood's Machine:
The two pulleys in the diagram shown below are identical and are uniform disks of mass m. Find an expression for the linear acceleration of the 4m mass.



is equivalent to



First apply Newton's second law to the three objects assuming that the acceleration of the 2m mass is equal in magnitude and opposite in direction to that of the 4m mass.

$$-2ma = T_1 - 2mg$$
$$4ma = T_2 - 4mg$$
$$I\alpha = T_1 R - T_2 R$$

The last equation needs help by expressing *I* and α in terms of *m* and *a,* then simplifying.

$$\frac{1}{2}(2mR^2)\frac{a}{R} = (T_1 - T_2)R$$
$$ma = (T_1 - T_2)$$

We have 3 equations and 3 unknowns. Eliminate the *T*'s and solve for *a*:

$$-6ma = (T_1 - T_2) + 2mg$$
$$-6ma = ma + 2mg$$

$$a = -\frac{2}{7}g.$$