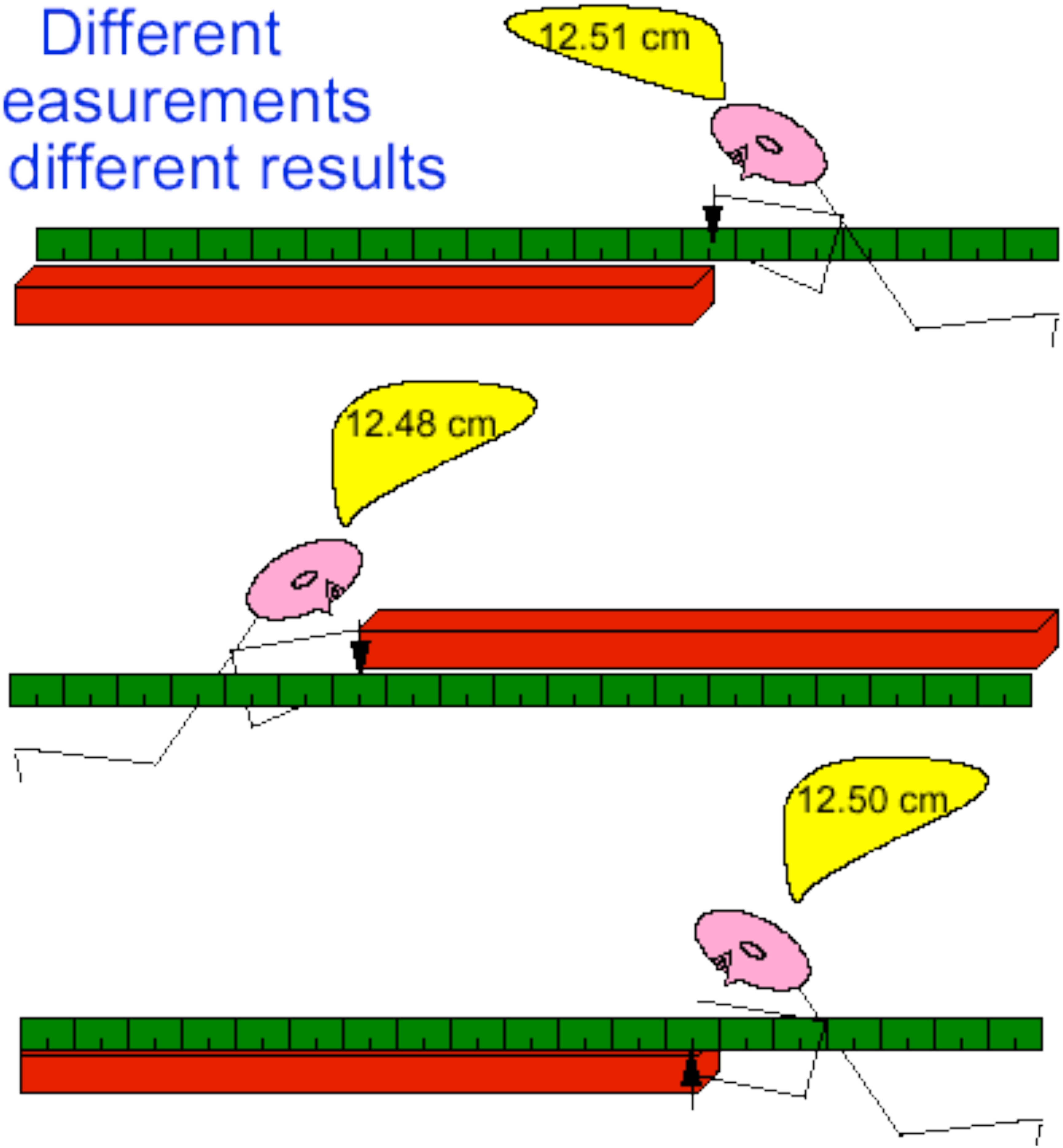
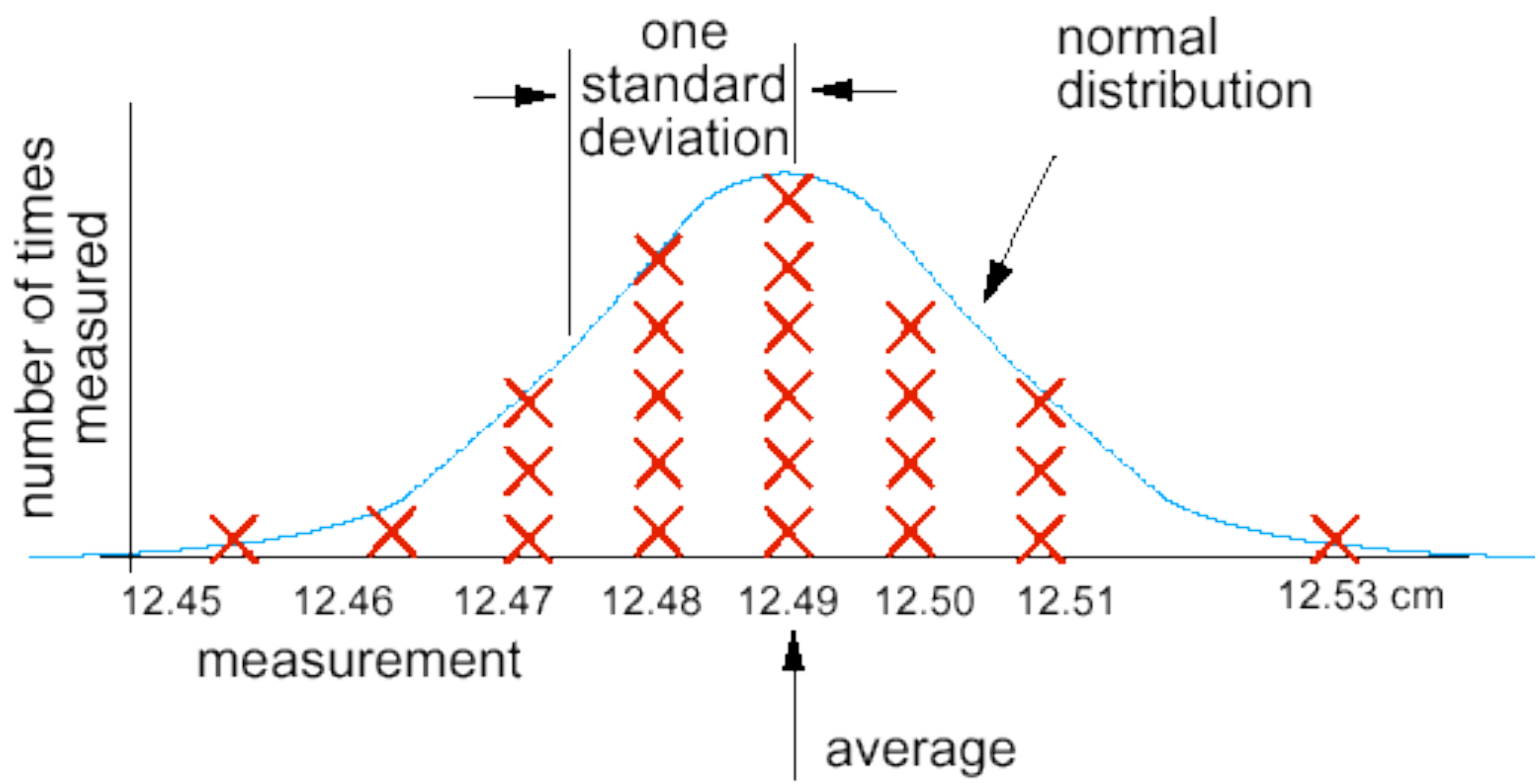
# How to deal with Uncertainty

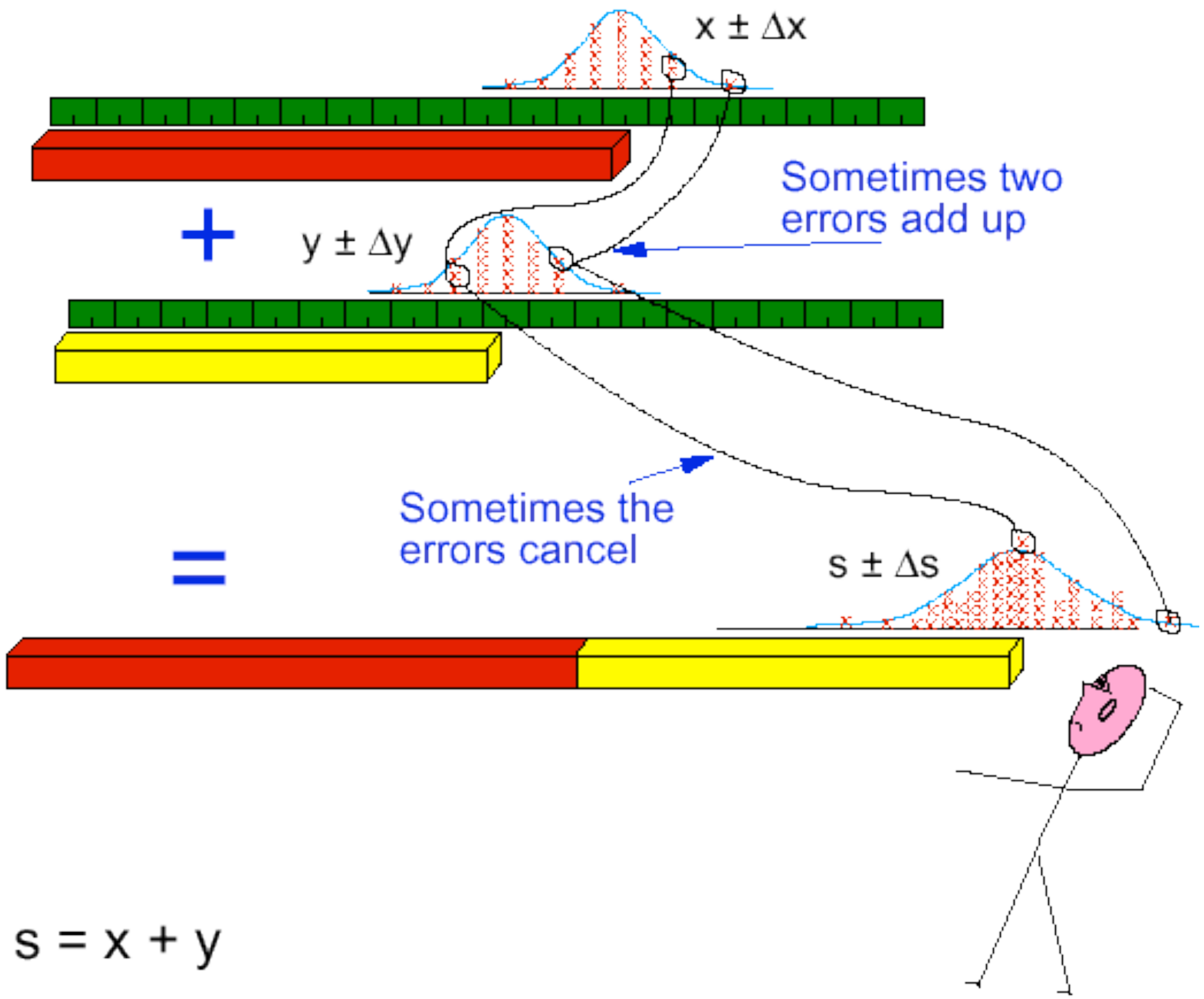
# Different measurements give different results



# After many measurements...



When you add two measurements which have random errors

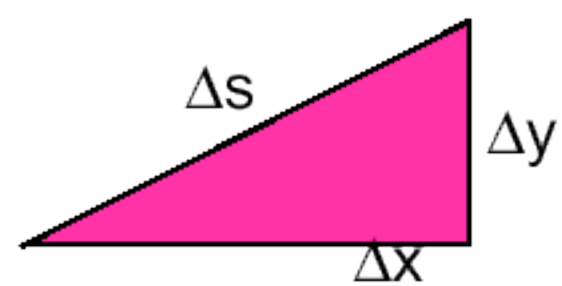



$s = x + y$

The error ranges of $x$ and $y$ are $\Delta x$ and $\Delta y$

The error range of $s$ is given by

$$\Delta s = \sqrt{\Delta x^2 + \Delta y^2}$$



# Propagation of Errors

Sum of two measurements

width $w$ = 0.24 ± 0.03 m
length $l$ = 0.89 ± 0.04 m

sum $s = w + l$ = 0.24 + 0.89 m = 1.13 m

possible error of the sum is

$$\Delta s = \sqrt{\Delta w^2 + \Delta l^2} = \sqrt{0.03^2 + 0.04^2} = 0.05 \text{ m}$$

The perimeter is twice the sum.

$$p = 2(w + l) = 2s = 2.26 \text{ m}$$

possible error of perimeter is

$$\Delta p = 2\Delta s = 0.1 \text{ m}$$

In conclusion write the perimeter as

$$\boxed{p = 2.3 \pm 0.1 \text{ m}}$$

Product of two measurements

$$\text{width } w = 0.24 \pm 0.03 \text{ m} = \frac{0.03}{0.24} = 12.5\ \%$$

$$\text{length } l = 0.89 \pm 0.04 \text{ m} = \frac{0.04}{0.89} = 4.5\ \%$$

Area is the product of width and length

$$A = wl = (0.24 \text{ m})\ (0.89 \text{ m}) = 0.2136 \text{ m}^2$$

Possible error of area is given by

$$\frac{\Delta A}{A} = \sqrt{\left(\frac{\Delta w}{w}\right)^2 + \left(\frac{\Delta l}{l}\right)^2}$$

(add percentage errors)

$$\frac{\Delta A}{A} = \sqrt{12.5\%^2 + 4.5\%^2} = 13\ \%$$

$$\Delta A = (0.13)(0.2136 \text{ m}^2) = 0.028 \text{ m}^2$$

So one should write

$$\boxed{A = 0.21 \pm 0.03 \text{ m}^2}$$

What do you do with wierd functions?

For example what is the possible error of

$$x = \cos(\theta)$$

when $\theta = 21° \pm 2°$ ?

$$x = \cos(21°) = 0.9335$$

The easiest way to find $\Delta x$ is to substitute for the minimum and maximum values:

$$\Delta\cos(\theta) = \frac{1}{2} \left| \cos(23°) - \cos(19°) \right|$$

$$= \frac{1}{2} \left| 0.9205 - 0.9455 \right| = \frac{0.024}{2}$$

$$= 0.012$$

so write

$$\boxed{x = 0.934 \pm 0.012}$$

Using Calculus to find $\Delta\cos(\theta)$:

$$\Delta x = \Delta\cos(\theta) = \left| \frac{d\cos(\theta)}{d\theta} \Delta\theta \right|$$

**Caution:**
If you use calculus, $\Delta\theta$ must be in **radians.**

$$\pm 2^\circ = \pm 2\frac{\pi}{180} = 0.035 \text{ radians}$$

$$21^\circ = 0.367 \text{ radians}$$

$$\Delta x = \Delta\cos(\theta) = |\sin(0.367)\,(0.035)|$$

$$= |(0.359)(0.035)|$$

$$= 0.0125$$

write 0.012 or 0.013 as you wish

**Question:**
This is slightly different from the substitution result of 0.12. Which is more accurate?

# Exponents

If a number is taken to a power, for example $t^2$, then the rule is to multiply the relative error by the power:

$t = 1.25 \pm 0.01$ s, relative error $\approx$ 0.8%

$t^2 = 1.56$ s, relative error $\approx$ 1.6%

so the result would be written

$$t^2 = 1.56 \pm 0.03 \text{ s}$$

(One could possibly write $1.56 \pm 0.025$ s in this case.)

The relative error of the square root of a value is one-half that of the value — it gets smaller.

In general for any power α

$$\frac{\Delta(x^{\alpha})}{x^{\alpha}} = \alpha \frac{\Delta x}{x}$$

# The fine print

## 0.6 Propagation of Uncertainties: General Rules

The uncertainty of calculated quantities depends directly on the uncertainties of the variables used in the calculation. For brevity we simply state the rules for commonly encountered situations here. Later some of these rules will be justified but a complete understanding needs statistical methods which are too advanced for this course.

In the following let $A, B, C$, ... stand for *independent* quantities going into a calculation with uncertainties $\Delta A$, $\Delta B$, $\Delta C$, .... Let $Y = f(A,B,C, ...)$ be the calculated quantity of interest.

### 0.6.1 **Rule 1: A constant multiple**

If

$$Y = k\,A$$

where $k$ is a constant, then

$$\Delta Y = k\,\Delta A$$

### 0.6.2 **Rule 2: Addition and Subtraction**

If

$$Y = A \pm B \pm C$$

then

$$\Delta Y = \sqrt{(\Delta A)^2 + (\Delta B)^2 + (\Delta C)^2}$$

The generalization to four or more addends should be obvious. The reason for taking the root-squared sum instead of just adding the uncertainties is that we are not certain whether the errors will cancel or add. If there are many terms in the sum, there will typically be some cancellation and the combined error will not likely be as large as the error given by the sum $|\Delta A| + |\Delta B| + |\Delta C| + ...$ .

### 0.6.3 **Rule 3: Multiplication and Division**

If

$$Y = ABC,\ Y = ABC^{-1},\ Y = AB^{-1}C^{-1}, \text{ or } Y = A^{-1}B^{-1}C^{-1}$$

then

$$\frac{\Delta Y}{Y} = \sqrt{\left(\frac{\Delta A}{A}\right)^2 + \left(\frac{\Delta B}{B}\right)^2 + \left(\frac{\Delta C}{C}\right)^2}$$

For multiplication and division we add the fractional (or percentage) errors.

### 0.6.4 **Rule 4: Powers**

If

$Y = A^{\alpha}$, where $\alpha$ is arbitrary: integer, fraction, positive or negative

then

$$\frac{\Delta Y}{Y} = |\alpha|\frac{\Delta A}{A}$$

### 0.6.5 **Examples**

(1) $Y = AB^2$

Let $C = B^2$ so that $Y = AC$.

Then according to rule 3 $\quad \dfrac{\Delta Y}{Y} = \sqrt{\left(\dfrac{\Delta A}{A}\right)^2 + \left(\dfrac{\Delta C}{C}\right)^2}$ .

According to rule 4 $\quad \dfrac{\Delta C}{C} = 2\dfrac{\Delta B}{B}$

Therefore $\quad \dfrac{\Delta Y}{Y} = \sqrt{\left(\dfrac{\Delta A}{A}\right)^2 + 4\left(\dfrac{\Delta B}{B}\right)^2}$ .

Example (2) $Y = \frac{1}{A} + \frac{1}{B}$

Let $C = A^{-1}$ and $D = B^{-1}$.

According to rule 4 $\frac{\Delta A}{A} = \frac{\Delta C}{C}$ and $\frac{\Delta B}{B} = \frac{\Delta D}{D}$.

However $C = A^{-1}$ and $D = B^{-1}$ so these latter two expressions can be written as

$$\Delta C = \Delta A/A^2 \text{ and } \Delta D = \Delta B/B^2.$$

Hence

$$\Delta Y = \sqrt{\left(\frac{\Delta A}{A^2}\right)^2 + \left(\frac{\Delta B}{B^2}\right)^2}.$$

0.6.6 **The General Case**

The four rules and the rule for the general case can be derived with the help of calculus. In this discussion we will assume a function of the form

$$Y = f(A,B) \tag{1}$$

Generalization to functions of more variables is easy.

Calculus tells us that if we change $A$ by a small amount $dA$ and $B$ by $dB$ then the change in $Y$ is give by

$$dY = \left(\frac{\partial f}{\partial A}\right)_B dA + \left(\frac{\partial f}{\partial B}\right)_A dB \tag{2}$$

where the subscript $A$ or $B$ on the partial derivatives has the conventional meaning that the quantity $A$ or $B$ is to be held fixed while taking the derivative. (That is the definition of partial derivative.) For convenience these subscripts are omitted from now on.

This equation tells us how fast the function $Y$ changes when we change its inputs $A$ and $B$ by some small amounts $dA$ and $dB$. We can identify these small changes with small errors in our measurements, $\pm\Delta A$ and $\pm\Delta B$. These errors can have either algebraic sign and so can the derivatives $(\partial f/\partial A)$ and $(\partial f/\partial B)$. In the worst case both terms in (2) are positive or both negative in which case you have

$$\Delta Y_{\text{worstcase}} = \left|\left(\frac{\partial f}{\partial A}\right)\Delta A\right| + \left|\left(\frac{\partial f}{\partial B}\right)\Delta B\right| \tag{3}$$

On the other hand it could turn out that you are lucky and the two terms in equation (2) tend to cancel. Then you would have

$$\Delta Y_{\text{bestcase}} = \left|\left|\left(\frac{\partial f}{\partial A}\right)\Delta A\right| - \left|\left(\frac{\partial f}{\partial B}\right)\Delta B\right|\right| \tag{4}$$

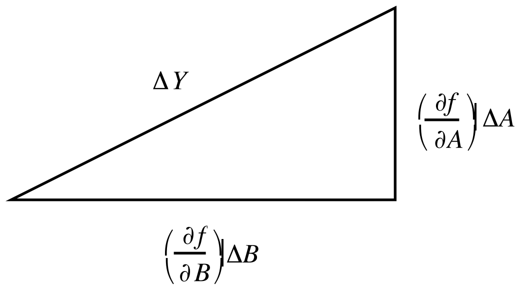
In practice there is no way of knowing if the errors are going to cancel or add in the final answer. Probability theory says that in this case, if the errors are independent and have a normal distribution then we should add the individual errors "in quadrature," i.e., form the root-squared sum as follows

$$\Delta Y = \sqrt{\left(\left(\frac{\partial f}{\partial A}\right)\Delta A\right)^2 + \left(\left(\frac{\partial f}{\partial B}\right)\Delta B\right)^2} \tag{5}$$

Note that $\Delta Y$ has the property

$$\Delta Y_{\text{worst case}} \geq \Delta Y \geq \Delta Y_{\text{best case}}.$$

You can visualize this way of adding errors by means of a right triangle.




The generalization of equation (5) to an arbitrary number of variables $Y = f(A,B,C...)$ is

$$\Delta Y = \sqrt{\left(\left(\frac{\partial f}{\partial A}\right)\Delta A\right)^2 + \left(\left(\frac{\partial f}{\partial B}\right)\Delta B\right)^2 + \left(\left(\frac{\partial f}{\partial C}\right)\Delta C\right)^2 + \ldots} \tag{6}$$

With a little effort you should be able to convince yourself that rules 1 through 4 are but special cases of equation (5). Equation (5) can also be used where the rules do not apply such as trigonometric or exponential functions.