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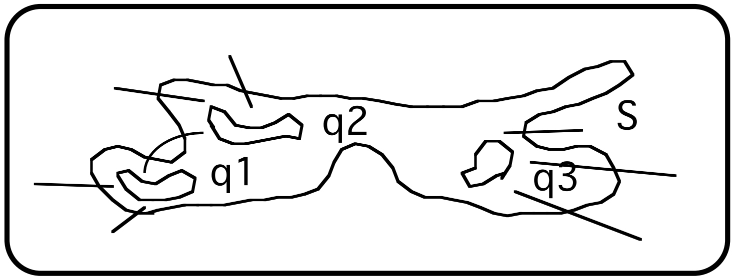
UNIT 20: ELECTRIC FLUX AND GAUSS' LAW



. . . before Maxwell people considered physical reality
. . . as material points . . . After Maxwell they considered
physical reality as continuous fields. . .

-- A. Einstein

OBJECTIVES

1. To understand how electric field lines and electric flux can be used to describe the magnitude and direction of the electric field in a small region in space.

2. To discover how the electric flux passing through a small area is related to the magnitude and direction of the area relative to the magnitude and direction of the electric field lines.

3. To discover the relationship between the flux passing through a "closed surface" and the charge enclosed by that surface for a two-dimensional situation (Gauss' law in Flatland).

4. To review how the expression over a closed three dimensional surface is proportional to the number of field lines passing through the closed surface and thus to extend the discovery of Gauss' law in Flatland to three dimensions.

5. To explore the concept of symmetry.

6. To learn to use Gauss' law to calculate the electric fields that result from highly symmetric distributions of electric charge at various points in space.

OVERVIEW

Coulomb's law and the principle of superposition can be used to calculate the force, and hence the electric field, on a test charge due to charge distributions that surround it. It is possible, however, to calculate the electric field using a completely different formulation of Coulomb's law. This formulation is known as Gauss' law and it involves relating the field surrounding a collection of charges to the amount of charge enclosed by a surface. The Gauss' law formulation is a very powerful tool for calculating electric fields due to *symmetric* distributions of charge. By using the rules of integration and vector algebra, Gauss' law can be proven to be mathematically equivalent to Coulomb's law.

You will begin the study of Gauss' law by learning about a convenient construct known as *electric field lines*. These lines can be used to map the direction of the net force on a small test charge at any point in space due to other charges. You will practice constructing the electric field lines from a configuration of charges using superposition and Coulomb's law.

Next you are going to take a non-mathematical approach to discovering a two-dimensional Gauss' law. You will examine patterns of electric field lines around collections of charges which were created for you using a computer simulation. You will then draw closed "surfaces" around various charges or groups of charges and see how many electric field lines pass in and out of the surfaces. Finally, you can explore the concept of symmetry and use the mathematical representation of Gauss' law in three dimensions to calculate the electric field at various points in space due to uniform charge distributions.

SESSION ONE: ELECTRIC FIELD LINES AND FLUX

Electric Field Lines

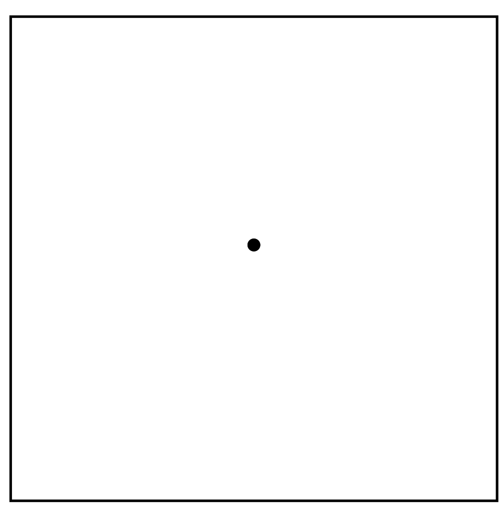
You have been representing the electric field due to a configuration of electric charges by an arrow that indicates magnitude and direction; using the principles of superposition and linearity, you can determine the length and direction of the arrow for each point in space. This is the *conventional representation* of a "vector field". An alternative representation of the vector field involves defining *electric field lines*. Unlike an electric field vector, which is an arrow with magnitude and direction, electric field lines are continuous. One can use a simulation to explore some of the properties of electric field lines for some simple situations.

For this activity you will need:

- Applet that simulates the direction and density of E-field lines. (http://www.sfu.ca/phys/141/applets/Efield_Lines.html)

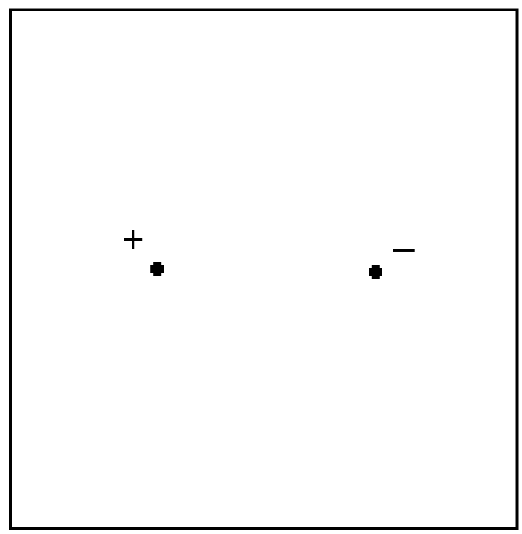
✍Activity 20-1: Simulated Electric Field Lines from Point Charges

(a) Put a single point charge on the screen and generate the electric field lines.. Sketch the electric field lines in the space below. Show the direction of the electric field on each line by placing arrows on them. Indicate how much charge you used on the diagram. How many lines are there in the drawing? Are the lines more dense or less dense near the charge? Explain. How does the direction of the lines depend on the sign of the charge?



(b) Try another image with a different magnitude of charge. **No need to sketch the result**, but how many lines are there? Can you describe the rule for telling how many lines will come out of or into a charge if you know the magnitude of the charge in Coulombs?

(c) Repeat the exercise for two charges with the *same magnitude* having *unlike* signs. Sketch the field lines in the space below. Indicate the size of the charges you used on the diagram. Comment on why the lines are more or less dense near the charges. How does the direction of the lines depend on the sign of the charge?

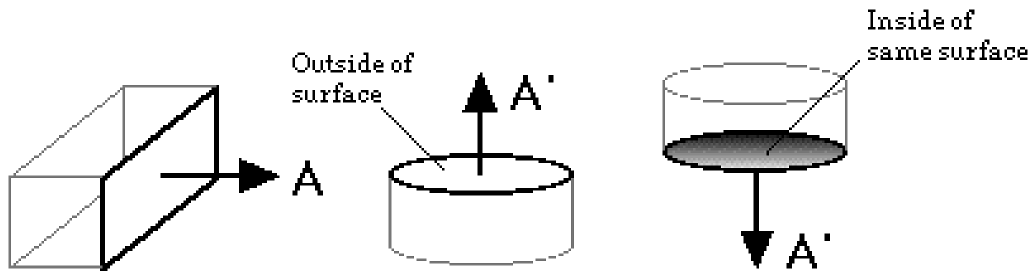


(d) Summarize the properties of electric field lines. What does the number of lines signify? What does the direction of a line at each point in space represent? What does the density of the lines reveal?

Electric Flux

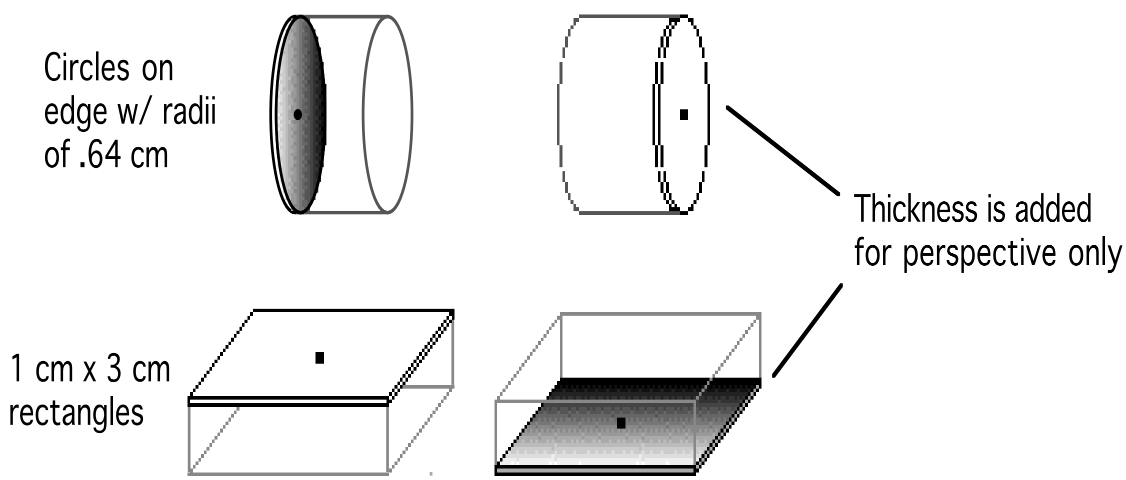
As you have just seen, we can think of an electrical charge as having a number of electric field lines, either converging on it or diverging from it, that is proportional to the magnitude of its charge. We can now explore the mathematics of enclosing charges with surfaces and seeing how many electric field lines pass through a given surface. *Electric flux* is defined as a measure of the number of electric field lines passing through a surface. In defining "flux" we are constructing a mental model of lines streaming out from the surface area surrounding each unit of charge like streams of water or rays of light. Modern physicists do not really think of charges as having anything real streaming out from them, but the mathematics that best describes the forces between charges is the same as the mathematics that describes streams of water or rays of light. So for now, let's explore the behavior of this model.

It should be obvious that the number of field lines passing through a surface depends on how that surface is oriented relative to the lines. The orientation of a small surface of area A is usually defined as a vector which is perpendicular to the surface and has a magnitude equal to the surface area. By convention, the normal vector points away from the *outside* of the surface. The normal vector is pictured below for two small surfaces of area A and A' respectively. In the picture it is assumed that the outside of each surface is white and the inside grey.



✍Activity 20-2: Drawing Normal Vectors

Use the definition of "normal to an area" given above to draw normals to the surfaces shown below. Let the length of the normal vector in cm be *equal in magnitude to each area in cm*^{2}. Don't just draw arrows of arbitrary length.

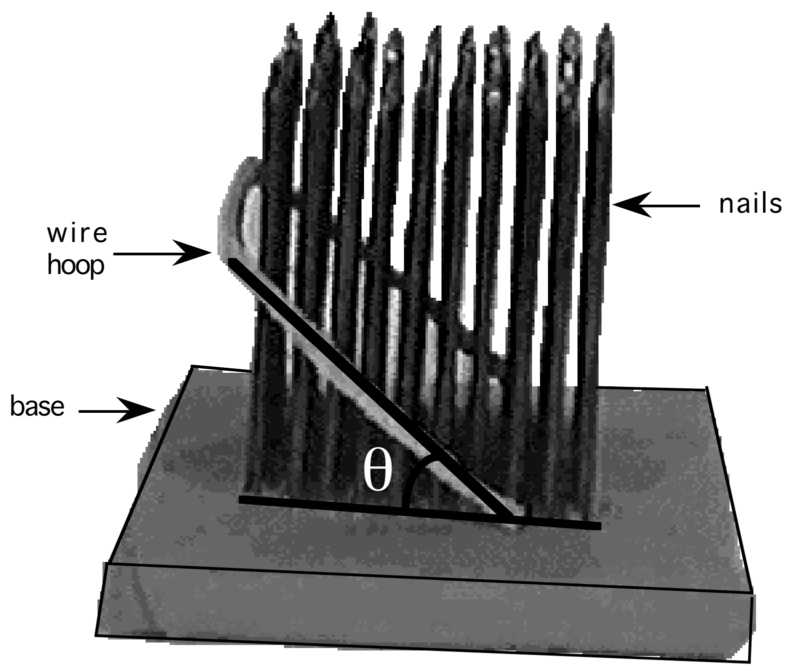


By convention, if an electric field line passes from the inside to the outside of a surface we say the flux is positive. If the field line passes from the outside to the inside of a surface, the flux is negative.

How does the flux through a surface depend on the angle between the normal vector to the surface and the electric field lines? In order to answer this question in a concrete way, you can look at a mechanical model of some electric field lines and of a surface. What happens to the electric flux as you rotate the surface at various angles between 0 degrees (or 0 radians) and 180 degrees (or π radians) with respect to the electric field vectors? The model consists of nails arranged in a 10 X 10 array poking up at 1/4" intervals through a board. The surface can be a copper loop painted white on the "outside". You will need:

- 100 nails or rods mounted on a square of Styrofoam or plywood
- 1 square wire loop
- 1 protractor

You can perform the measurements with a protractor and enter the angle in radians and the flux into a computer data table for graphical analysis.



✍Activity 20-3: Flux as a Function of Surface Angle

(a) Use your mechanical model, a protractor, and some calculations to fill in the data table below.

θ (°)	θ (rad)	Φ (# lines)
		100
		90
		80
		70
		60
		50
		40
		30
		20
		10
		0
		-10
		-20
		-30
		-40
		-50
		-60
		-70
		-80
		-90
		-100

Hint: You can use symmetry to determine the angles corresponding to negative flux without making more measurements.

(b) Plot the flux Φ as a function of radians. Look *very* carefully at the data. Is it a line or a curve? What mathematical function might describe the relationship?

(c) Try to confirm your guess by constructing a spreadsheet model and overlay graph of the data and the mathematical function you think matches the data. (Use the Modeling Worksheet.) Submit your spreadsheet with graph online. Also summarize your procedures and conclusions below.

(d) What is the definition of the vector dot product of two vectors in terms of vector magnitudes and the angle θ between them? Can you relate the scalar value of the flux, Φ, to the dot product of the vectors $\vec{E}$ and $\vec{A}$?

A Mathematical Representation of Flux through a Surface

One convenient way to express the relationship between angle and flux for a uniform electric field is to use the dot product so that $\Phi = \vec{E} \cdot \vec{A}$. Flux is a scalar. If the electric field is not uniform or if the surface subtends different angles with respect to the electric field lines, then we must calculate the flux by breaking the surface into infinitely many infinitesimal areas so that $d\Phi = \vec{E} \cdot d\vec{A}$, and then taking the integral sum of all of the flux elements. This gives

$$\Phi = \int d\Phi = \int \vec{E} \cdot d\vec{A} \qquad \text{[flux through a surface]}$$

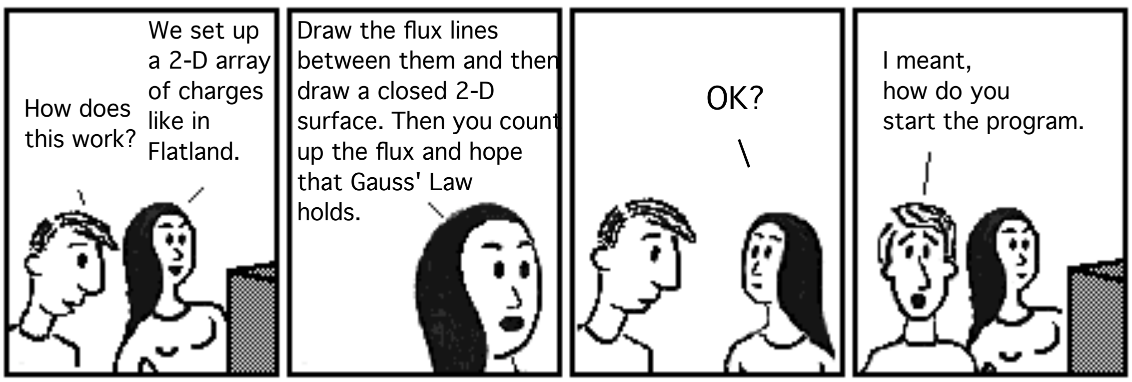
Some surfaces, like that of a sphere or that representing a rectangular box, are closed surfaces, i.e. they have no holes or breaks in them. Because we want to study the amount of flux passing through closed surfaces, there is a special notation to represent the integral of through a closed surface. It is represented as follows:

$$\Phi = \oint d\Phi = \oint \vec{E} \cdot d\vec{A}$$ [flux through a closed surface]

SESSION TWO: GAUSS' LAW

Discovering Gauss' Law in Flatland

How is the flux passing through a closed surface related to the enclosed charge? Let's pretend we live in a two-dimensional world in which all charges and electric field lines are constrained to lie in a flat 2 dimensional space – of course, mathematicians call such a space a plane*. You are going to examine some computer-generated images of more complicated charge configurations, in order to discover how the net number of lines passing through a surface is related to the net charge enclosed by the surface.



***Note:** If you haven't already read it, we recommend that you read E. A. Abbot's book entitled Flatland; a romance of many dimensions (Dover, New York, 1952). It's a delightful piece of late nineteenth century political satire in the guise of a mathematical spoof.

- Applet that simulates the direction and density of E-field lines. (http://www.sfu.ca/phys/141/applets/Efield_Lines.html

✍Activity 20-4: Gauss' Law in Flatland

(a) Create, using the applet, an image showing some positive *and* negative charges and the associated "E-field" lines. Draw arrows on each of the lines indicating in what direction a *small* positive test charge would move along each line.
Note: "Small" means that the test charge does not exert enough force on the charge distribution that creates the E-field to cause the field to change noticeably.

(b) Draw at least **three** two-dimensional closed "surfaces" ion your image. Some of them should enclose charge and some should avoid enclosing charge. Count the *net* number of lines of flux coming out of each "surface". Record your results in the table below.
Note: Consider each line coming out of a surface as positive and each line going into a surface as negative. The *net number of lines* is defined as the number of positive lines minus the number of negative lines. Φ

	Charge enclosed by the arbitrary surface			Lines of flux in and out of the surface		
	$+q$	$-q$	q_{net}	Φ_{out}	Φ_{in}	Φ_{net}
1						
2						
3						

(c) What is the apparent relationship between the net flux and the net charge enclosed by a two dimensional "surface"?

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Gauss' Law in Three Dimensions

If you were to repeat the exploration you just performed in a three-dimensional space, what do you think would be the appropriate expression for Gauss' law?

✍Activity 20-5: Statement of Gauss' Law

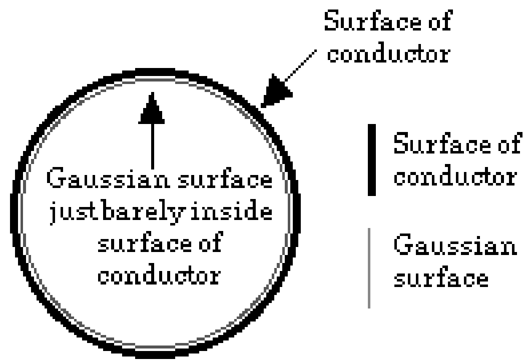
(a) Express the three-dimensional form of Gauss' law in words.

(b) Express the law using an equation.

Electric Fields and Charges Inside a Conductor

An electrical conductor is a material that has electrical charges in it that are free to move. If a charge in a conductor experiences an electric field it will move under the influence of that field since it is not bound (as it would be in an insulator). Thus, we can conclude that if there are no moving charges inside a conductor, the electric field in the conductor must be zero.

Let's consider a conductor that has been touched by a charged rubber rod so that it has an excess of negative charge on it. Where does this charge go if it is free to move? Is it distributed uniformly throughout the conductor? If we know that $\vec{E}=0$ inside a conductor we can use Gauss' law to figure out where the excess charge on the conductor is located.

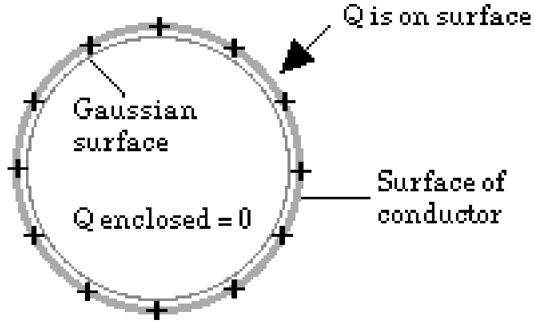


✍Activity 20-6: Where is the Excess Charge in a Metal?

(a) Consider a conductor with an excess charge of Q. If there is no electric field inside the conductor (by definition), then what is the amount of excess charge enclosed by the Gaussian surface just inside the surface of the conductor?
Hint: Use the three dimensional Gauss' law here.

(b) If the conductor has excess charge and it can't be inside the Gaussian surface according to Gauss' law, then what's the only place the charge can be?

(c) Forgetting for the moment about Gauss' law, but recalling that like charges repel each other, is the conclusion you reached in part (b) above physically reasonable? Explain. **Hint**: How can each unit of excess charge which is repelling every other unit of excess charge get as far away as possible from the other excess charges on the conductor?



Faraday's Cage: An Important Application of Gauss' Law

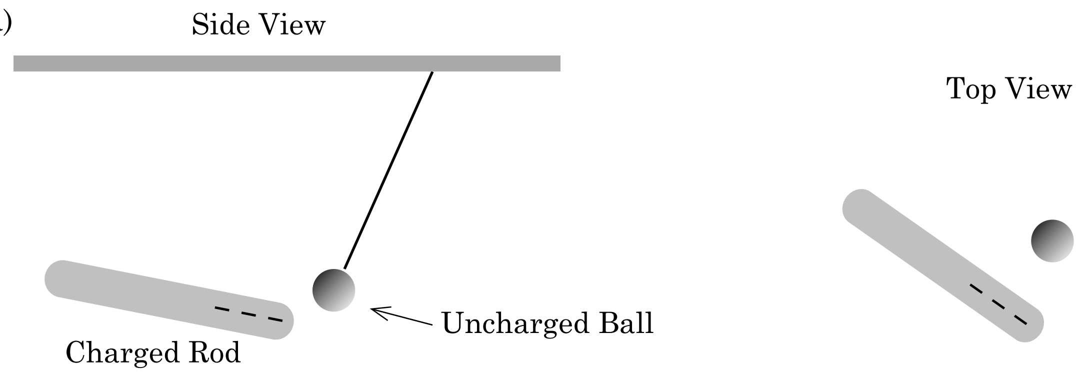
Gauss' Law has an important practical application. Imagine a solid conducting object which may be in a strong electric field. If all excess charge is on the surface of the conductor and if there is no electric field inside the conductor, then it should be possible to remove any conducting material from the interior without changing the location of the charges or the electric field inside. This will leave a hollow metallic shell which has no electric field inside it. There is no electric field inside even if there is a strong field outside: the conducting shell acts as a "shield" against electric fields. Furthermore, the hollow shell can have holes in it or even be made of a wire mesh and still retain its shielding property. Such a space surrounded by a conductor is called a "Faraday cage". You can test the Faraday cage with the following

- A metal cup
- A hard plastic rod and fur
- A metallic ball suspended by a thread

✍Activity 20-7: Experimental Verification of Faraday's Cage

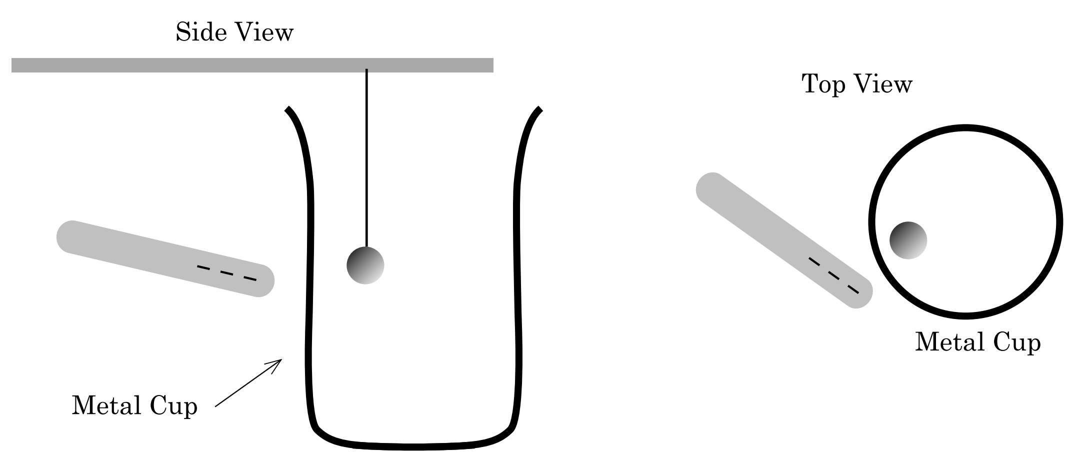
(a) Place the charged rod near to the uncharged ball and verify that the ball is attracted to the rod. The picture here shows a diagram of the side view and top view of the uncharged ball attracted to a charged rod. If the uncharged ball touches the rod then touch the ball with your finger to drain any acquired charge before you continue.

a)



(b) Hold the metal cup with your hand and place around the ball with the ball near but not touching the metal. If the ball is far below the lip of the cup then the metal cup effectively shields the ball from the electric field created by the rod. Try to attract the ball with a charged rod outside the cup. Does the electric field of the rod reach inside the metal cup?

b)



(c) Hold the rod in its position outside the cup as near as possible to the ball inside the cup. Slowly lower the cup and remove it, leaving the distance between the rod and ball about the same. What happens when the cup is removed?

Explain in your own pictures and words how the metallic cup manages to shield the ball from the electric field of the rod.

(d) Suppose that the cup is used to shield the ball while not being allowed to contact your bare hand or other electrical conductor. (Perhaps you wear a rubber glove or hold it with a nonconductor.) Will the shielding effect be the same? Before you try it predict what will happen and explain why.

(e) Try the experiment without touching the can with your hand or any other electrical conductor. Was your prediction verified or not?

(f) If you can locate a wire cage, try the experiment with the cage to see if the cage also exhibits the same shielding effect as the metal cup and report your results here.

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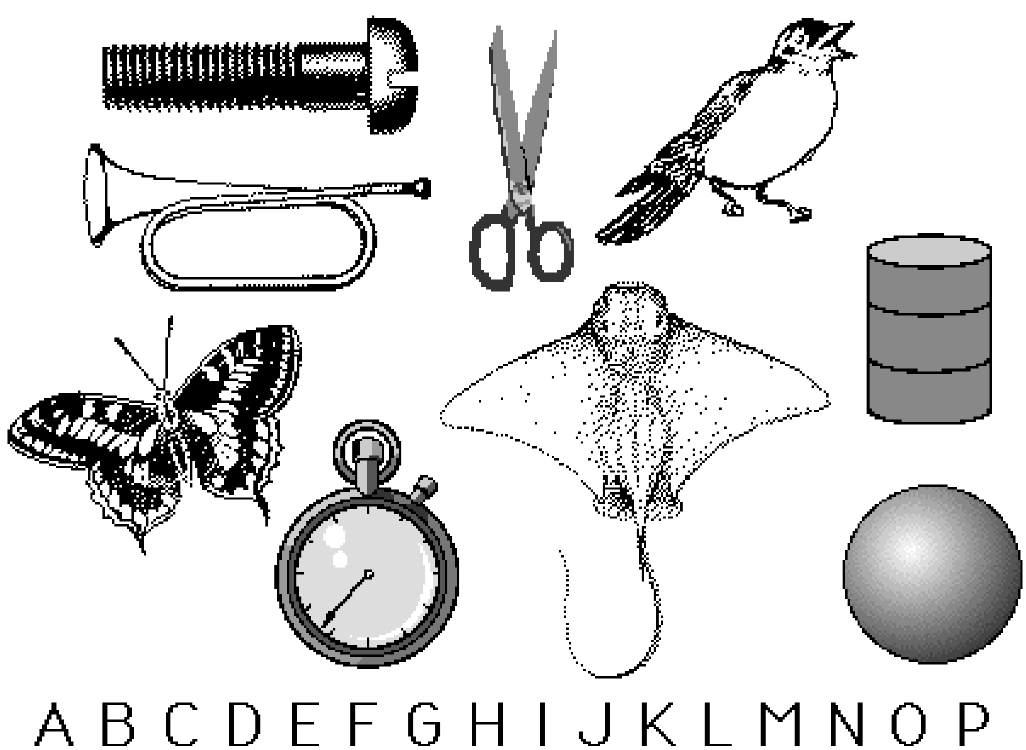
SESSION THREE: GAUSS' LAW CALCULATIONS

Symmetry

We're making all this fuss about Gauss' law so that you can calculate the electric field from *symmetric* charge distributions. What is a "symmetric" distribution? It's an arrangement of charges that can be rotated about an axis and/or reflected in a mirror and still look the same. Let's pretend that the objects below are made of rigid insulating materials and that charge is uniformly distributed throughout each object. Based on the *external* appearance of the objects, which ones appear to have symmetry? Which objects don't appear symmetric? Don't worry about the internal structure of the objects. Some of these situations are subtle (e.g. does the mirror image of a screw turn the same way?).

✍Activity 20-8: Symmetric Charge Distributions

Circle the symmetric charge distributions. Don't forget to look at the letters!



Using Gauss' Law to Calculate Electric Fields

Gauss' law can be stated mathematically by means of the expression

$$\Phi = \oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_o}$$

where ε_o is the *permittivity of free space* ($\varepsilon_o = 8.85 \times 10^{-12}\ C^2/(Nm^2)$). Gauss' law is typically used to compute the electric field at some distance from a uniform charge distribution that is symmetric. We will be interested in two types of symmetry – cylindrical symmetry and spherical symmetry. Learning to apply Gauss' law will take some practice. The key is to pick a closed surface that has the same symmetry as the charge distribution causing the electric field. Thus, a spherical surface works for point charges and spherical distributions and a cylindrical surface works for line charge distributions and cylindrical distributions.

Before we wade into the use of Gauss' law, lets review some geometry for some fairly simple symmetric shapes.

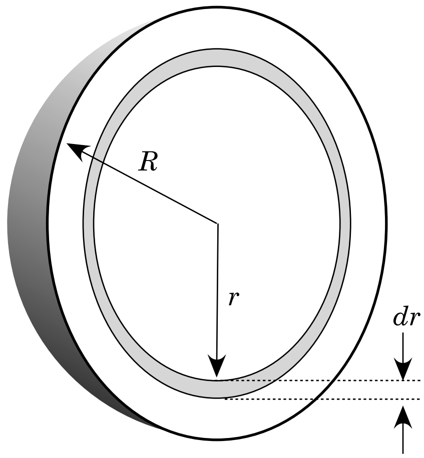
✍Activity 20-9: Some Geometry of Circles and Spheres

(a) What is the equation for the circumference of a circle of radius *r*? If the radius doubles what happens to the circumference?

(b) What is the equation for the area of a circle of radius *r*? If the radius doubles what happens to the area?

(c) What is the equation for the volume of a sphere of radius *r*? If the radius doubles what happens to the volume?

(d) Find the derivative of the volume V of a sphere as a function of r. Show how this derivative can be used to determine how much the volume of a sphere would increase (i.e. the factor dV) if the radius of the sphere were increased from r to $r + dr$.
Hint: Consider this increase as being the volume of a shell of thickness dr surrounding the sphere.



(e) If the letter S is used to represent the surface area of a sphere, what is the volume dV of a thin shell of thickness dr that surrounds a sphere of radius r in terms of S and dr?

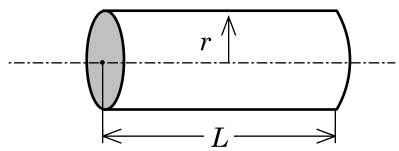
(f) Use the derivative dV/dr from part (d) and the idea that a spherical shell represents a volume increase to show that the surface area of a sphere can be represented by the equation
$S= 4\pi r^2$.

(g) If the radius of a sphere doubles how much does its surface area increase?

(h) If a charge Q is spread uniformly throughout the volume of an *insulating* sphere (i.e. charge cannot move around inside it) of radius r, what fraction of the charge lies within a radius of $r/2$?
Warning: The answer is <u>not</u> 1/2.

✍Activity 20-10: Some Geometry of Cylinders

(a) Consider a cylinder of radius r and length L. What is its volume in terms of π, r, and L?



(b) If a charge Q is spread uniformly throughout the volume of a cylinder of radius r and length L, what fraction of the charge lies within a radius of $r/2$? **Warning**: The answer is *not* 1/2.

(c) What is the surface area of the cylinder in terms of π, r, and L?. **Hint**: Don't neglect the ends.

After this geometry review, you should now be warmed up enough to tackle the problem of how to use Gauss' law to find the electric field at a distance r from a point charge. You should also be able to find the electric field at a distance r from an insulator with a uniform, spherically symmetric charge distribution. This is a standard derivation so you can consult any introductory physics text for hints if necessary. Before you get started, let's take a quick look at charge density.

The concept of *charge density* is very useful in figuring out how much charge is contained within a given radius r in a charged sphere. If a volume element dV contains a small amount of charge dq then the charge density is given by the equation:

$$\rho = \frac{dq}{dV}$$

Thus, if the charge is distributed in a spherically symmetric manner, the amount of charge contained within a radius r is given by:

$$q = \int_0^r \rho\, dV = \int_0^r \rho\, 4\pi\, r^2 dr = 4\pi \int_0^r \rho\, r^2 dr$$

In the case where the charge is uniformly distributed, ρ is a constant given by:

$$\rho = \frac{Q}{\frac{4}{3}\pi R^3}$$

where Q is the total charge in the sphere and R is the sphere's radius. In this case, then, the charge contained within a radius r is given by:

$$q = \int_0^r \rho\, dV = 4\pi\, \rho \int_0^r r^2 dr$$

✍Activity 20-11: Gauss' Law & Spherically Symmetric Charge Distributions

(a) Use Gauss' law to calculate the electric field at a distance r from a point charge $+q$. **Hints**: (1) If you use a spherical shell of radius r as your closed surface, then by symmetry the magnitude of the electric field is the same at all points on the sphere, and thus E can be factored out of the integral. (2) Since $4\pi r^2$ is the equation for the area of the surface of a sphere, $\oint d\vec{A} = 4\pi r^2$.

(b) Compute the magnitude of the electric field at a distance r from the centre of a uniformly charged sphere of radius R with a total charge of Q throughout its volume, where $r < R$. (I.e., the point in question is inside the sphere!) Start by using the equation for the volume of a sphere of radius r ($V = \frac{4}{3}\pi r^3$) to show that the charge q that lies within a radius r is given by

$$q = Q\frac{r^3}{R^3}$$

(c) Next, use Gauss' law along with the equation you just derived to show that the magnitude of the electric field
inside a uniformly charged sphere of radius R having a total charge of Q is

$$E = \left|\vec{E}\right| = k\frac{Qr}{R^3} \quad \text{where } k = \frac{1}{4\pi\varepsilon_0}$$

What happens to the electric field when there is cylindrical symmetry? Can Gauss' law be used to find the electric field at a distance r from an infinite line of electrical charges? How about the electric field at a distance r from an *infinitely* long insulator with a uniform charge distribution that has cylindrical symmetry? Consult an introductory text for hints if necessary

✍Activity 20-12: Gauss' Law and Cylindrical Symmetry

(a) Use Gauss' law to calculate the electric field at a distance *r* from a very long, straight, uniformly charged cylinder that has a charge per unit length of λ. (Mathematically, we can treat a wire which is *physically* very long as if it were infinitely long.) **Hint**: Using a symmetry argument, explain why you can neglect the electric field perpendicular to the two ends of the cylinder.

(b) Assume that the charge per unit length $\lambda = 5.00\text{x}10^{-9}$ C/m. Plug some numbers into the equation you derived to find the magnitude of the electric field as a distance of 5.00 cm from the line of charge As usual you should express your result using the appropriate number of significant figures.

$\varepsilon o = 8.85 \text{ x } 10^{-12}\ C^2/(N \cdot m^2)$
$k = 1/4\pi\varepsilon o = 8.99 \text{ x } 10^{9}\ (N \cdot m^2)/C^2$

(c) What is the *direction* of the electric field? Explain the reasons for your answer.

(d) How do we know that the E-field lines are perpendicular to the wire (i.e. that they have no component which is parallel to the wire)?