Proportional feedback control of chaos in a simple electronic oscillator

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We demonstrate the control of chaos in a nonlinear circuit constructed from readily available electronic components. Control is achieved using recursive proportional feedback, which is applicable to chaotic dynamics in highly dissipative systems and can be implemented using experimental data in the absence of model equations. The application of recursive proportional feedback to a simple electronic oscillator provides an undergraduate laboratory problem for exploring proportional feedback algorithms used to control chaos.

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I. INTRODUCTION

Chaos theory is the study of deterministic systems whose dynamics are aperiodic and depend sensitively on initial conditions.[1] Chaotic systems have long term behavior that is unpredictable. Nonlinear dynamical systems often behave chaotically for certain ranges of system parameters; for other parameter ranges they behave periodically and thus predictably. Proportional feedback control of chaos involves perturbing a system parameter, while maintaining its value within its normally chaotic range, to achieve stabilization of a selected trajectory on the system's chaotic attractor. In this paper we focus on a proportional feedback control strategy that stabilizes periodic dynamics.

In 1990, Ott, Grebogi, and Yorke (OGY)[2] introduced a proportional feedback algorithm suitable for a large class of nonlinear oscillators. Their approach employs a feedback loop that applies small perturbations to a system at the end of each oscillation, with each perturbation proportional to the difference between the current state and a desired state. This strategy is an extension of engineering control theory.[3,4] The OGY algorithm precipitated an outpouring of experimental and theoretical work on controlling chaotic dynamics.[5] Proportional feedback control has been demonstrated for a wide variety of dynamical systems including mechanical,[6,7] fluid,[8] electronic,[9] optical,[10] chemical,[11,12] and biological[13] systems. Control via proportional feedback is now a central topic of research in nonlinear dynamics and has been extended experimentally to chaotic spatial patterns.[14,15] Given its currency and prominence, it is desirable to introduce this topic in undergraduate courses on chaos.

Baker[16] has provided a clear presentation of the OGY algorithm. However, the algorithm is challenging to implement experimentally because it requires sampling more than one dynamical variable in real time. Dressler and Nitsche[17] showed that the OGY algorithm can be modified to allow for measurements of a single dynamical variable. But there are much simpler alternatives to the OGY algorithm for highly dissipative systems, that is, systems whose dynamics can be reduced to one-dimensional (1D) return maps. For this special but not uncommon case, Peng, Petrov, and Showalter[18] introduced simple proportional feedback, and Rollins, Parmananda, and Sherard[19] derived recursive proportional feedback. These less complicated proportional-feedback algorithms illustrate key ideas of control and are considerably less mathematically demanding than the OGY algorithm, which is applicable to a wider range of systems.

In this paper we present a derivation of recursive proportional feedback at a level suitable for an introductory course on chaos, and we explain why recursive proportional feedback is more generally applicable than simple proportional feedback. Flynn and Wilson[20] and Corron, Pethel, and Hopper[21] have presented other simple methods of controlling chaos that are also suitable for introducing undergraduates to this topic. However, a discussion of simple proportional feedback and recursive proportional feedback allows us to address the issues of the stability and the range of applicability of individual control algorithms.

Recently, Kiers, Schmidt, and Sprott (KSS)[22] introduced a simple nonlinear electronic circuit that can be used to study chaotic phenomena. This circuit employs readily available electronic components and is well-suited for advanced undergraduate instructional laboratories. A novel feature of the KSS circuit is the presence of an almost ideal nonlinear element, which results in excellent agreement between the experimental circuit and numerical solutions of the differential equation that models the dynamics of the circuit. The circuit allows for precise measurements of bifurcation diagrams, phase portraits, return maps, power spectra, Lyapunov exponents, and the fractal dimension of chaotic attractors. A further advantage of using the KSS circuit for undergraduate experiments is that the time scale can be adjusted so that the periods of oscillation are on the order of a second, making the circuit an ultra-low-frequency electronic oscillator. Students can observe the dynamics in real time, and there is sufficient time during the oscillations for a digital processor to compute the requisite perturbations for chaos control. Undergraduates can readily wire the circuit, interface it to a computer-based data acquisition board, and write a program to acquire data and apply a proportional feedback loop. There are many data acquisition, output, and analysis systems in use in undergraduate laboratories that could be employed.

In this paper we show that the KSS circuit can be used to illustrate proportional feedback control of chaos by applying recursive proportional feedback to its dynamics. We show how chaos can be controlled experimentally, even in the absence of model equations, by determining the values of the coefficients in the recursive proportional feedback algorithm only from experimental data, without reference to the differ-

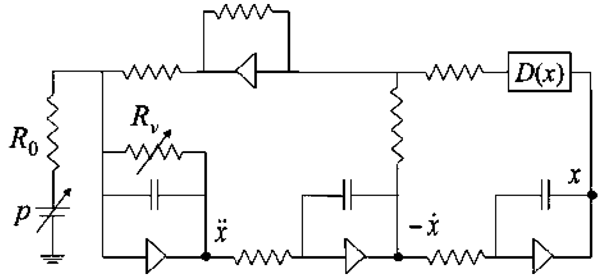

Fig. 1. Circuit diagram of the electronic oscillator modeled by Eq. (1). We used dual LMC6062 op amps, chosen for their high input impedance, throughout the circuit. The effects of noise on the circuit are reduced by capacitively tying the power supplies for the op amps to ground. The wiring into the op amps is to their inverting (−) inputs; the noninverting (+) inputs are grounded. The supply voltages for the op amps were set at ±15 V. We chose component values for which the circuit oscillates chaotically. The unmarked resistors and capacitors are $R = 46.6 \pm 0.3\ \mathrm{k}\Omega$ and $C = 2.29 \pm 0.03\ \mu\mathrm{F}$. The variable resistor $R_V = 81.4 \pm 0.1\ \mathrm{k}\Omega$ and $R_0 = 156.9 \pm 0.1\ \mathrm{k}\Omega$.

ential equation that models the dynamics of the circuit. However, because the model equation is known, an important feature of this circuit is that one can perform analytic and numerical investigations in parallel with the experiment. Our demonstration of control of a chaotic electronic oscillator fits well in an undergraduate course on nonlinear dynamics or computer interfaced experimentation.

II. CHAOS IN A SIMPLE ELECTRONIC OSCILLATOR

Jerk equations (third-order autonomous ordinary differential equations) with nonlinearities involving piecewise linear functions often can be implemented electronically using only resistors, capacitors, diodes, and op amps.[23] Many of these equations and corresponding electronic circuits exhibit chaotic oscillations for a range of system parameters. The KSS circuit is an example of such a chaotic electronic oscillator. Its diagram is shown in Fig. 1, and it can be assembled on a standard solderless breadboard. The circuit contains three successive inverting amplifiers with output voltages at the nodes labeled x, $-\dot{x}$, and $\ddot{x}$. These outputs are the dynamical variables of the system. The input voltage for the circuit, which is labeled p, is an accessible system parameter that can be varied. The box labeled $D(x)$ represents the nonlinear element in the circuit, which is necessary for chaotic oscillations.

By using Kirchhoff's rules and the golden rules for op amps, we obtain the following dynamical equation (see Ref. 22 for details):

$$\dddot{x} = -\left(\frac{R}{R_V}\right)\ddot{x} - \dot{x} + D(x) - \left(\frac{R}{R_0}\right)p, \tag{1}$$

where a dot denotes differentiation with respect to the dimensionless time scaled by RC. R_V is a variable resistor; varying R_V allows one to explore a wide range of the circuit's dynamical behavior.[22] To make explicit that there are three dynamical variables for this circuit (as well as to facilitate numerical integration using a Runge-Kutta method), we rewrite Eq. (1) as a system of three first-order autonomous ordinary differential equations. If we define $y \equiv \dot{x}$ and $z \equiv \ddot{x}$, then

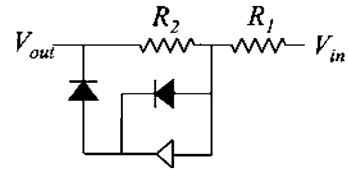

Fig. 2. The nonlinear subcircuit $D(x)$ corresponding to the box in Fig. 1. The wiring into the op amp is to its inverting (−) input; the noninverting (+) input is grounded and 1N914 diodes are used. The resistors have values $R_1 = 15.1 \pm 0.1\ \mathrm{k}\Omega$ and $R_2 = 88.9 \pm 0.1\ \mathrm{k}\Omega$ so that $R_2/R_1 \approx 6$.

$$\dot{x} = y, \tag{2a}$$

$$\dot{y} = z, \tag{2b}$$

$$\dot{z} = -\left(\frac{R}{R_V}\right)z - y + D(x) - \left(\frac{R}{R_0}\right)p. \tag{2c}$$

The nonlinearity in the KSS circuit is modeled by the function $D(x) = -(R_2/R_1)\min(x,0)$; Figure 2 shows the diagram for this nonlinear subcircuit. The agreement between the piecewise linear function $D(x)$ and the actual output of the subcircuit is excellent and leads to impressive agreement between measured values of the dynamical variables and numerical solutions of Eq. (2).[22]

One technique for representing chaotic dynamics is to construct return maps. For a first-iterate return map, a sequence of maximum values of a dynamical variable is used and x_{n+1} is plotted versus x_n. If this plot forms a thin, approximately 1D curve, then the dynamics have been reduced to a 1D map: $x_{n+1} = f(x_n, p_0)$, where p_0 is the value of the system parameter at which the sequence was collected. Figure 3 shows a 1D return map for the KSS circuit for a sequence of maxima of the output voltage x during chaotic oscillations. In practice there may be several other system parameters on which the mapping depends, as is the case for the KSS circuit (for example, R_V, R_0, R_1, R_2, and R). However, if these other parameters remain fixed throughout the experiment, they can be ignored. Systems that can be reduced to 1D maps are referred to as *highly dissipative*.[19] Examples of other experimental systems that exhibit highly dissipative chaotic dynamics include a gravitationally buck-

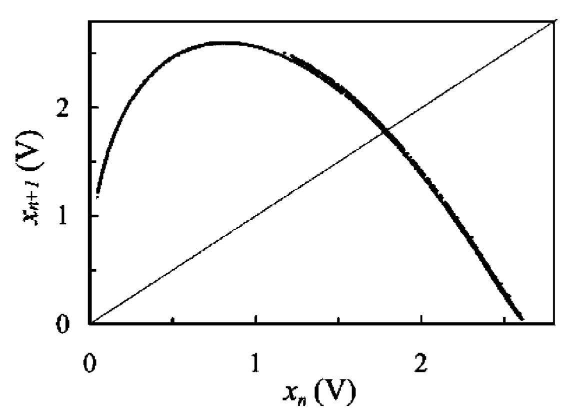

Fig. 3. First-iterate return map of an experimentally measured sequence of 1913 maxima of x, collected with the input voltage fixed at $p_0 = 1.0000$ V. The intersection of the map with the $x_{n+1} = x_n$ line corresponds to an unstable period-one fixed point.

led magnetoelastic ribbon,[6] Belousov-Zhabotinsky oscillating chemical reactions,[24] electrodissolution in an electrochemical cell,[11,12] electronic circuits,[25] and phase slips in a pattern of fluid vortices.[14]

III. DERIVATION OF THE PROPORTIONAL FEEDBACK CONTROL ALGORITHM

When a system is in a chaotic state, there typically are infinitely many unstable periodic orbits embedded in its dynamics. A fixed point of a return map corresponds to a periodic orbit. For example, a period-one orbit returns to the same value after one oscillation: $x_{n+1}=x_n$. Thus a period-one fixed point is defined as a value of the dynamical variable for which the map returns the same value: $x_F \equiv f(x_F, p_0)$. For systems undergoing chaotic dynamics, such fixed points are unstable, and the key to controlling chaos using proportional feedback algorithms is to stabilize a dynamical variable near an unstable fixed point. This stabilization can be accomplished by applying a sequence of small perturbations δp_n, one after each iteration of the map, to an accessible system parameter p on which the map depends. The strength of the applied perturbation is proportional to the difference between the current value of a dynamical variable and the predetermined fixed point: $\delta p_n \propto (x_n - x_F)$. The perturbations are limited in magnitude by the requirement that the adjusted parameter remain within a range for which the system is chaotic in the absence of perturbations. The control algorithm is a feedback loop that samples the variable in real time and adjusts the parameter accordingly. The carefully chosen perturbations alter the system in such a manner that the current state of the system will evolve closer to the fixed point during the next cycle of the perturbed system. This perturbing of the dynamics causes the otherwise chaotic oscillations to remain approximately periodic. To determine the proportionality constant in a control algorithm, it is necessary to identify a desired fixed point, determine how strongly unstable it is, and determine the response of the system to small changes in the parameter that will be perturbed to implement control. This procedure can be done using experimental measurements of the evolution of a single dynamical variable for various values of a system parameter prior to initiating control and requires no knowledge of an analytical model for the dynamics.

To derive the recursive proportional feedback algorithm, we consider nonlinear systems with three dynamical variables, the minimum necessary for chaos.[1] The KSS electronic oscillator is an example of such a system as can be seen from Eq. (2). For a system with three dynamical variables, x, y, and z, its chaotic attractor exists in a 3D phase space. If high dissipation limits the chaotic attractor to a very thin, nearly 2D surface, the dynamics are reducible to a 1D return map. This reduction is possible because the points on the 2D surface which intersect a particular surface of section, for example, the $y=y_c$ plane (where y_c is a constant), form a 1D curve. We may view the evolution of the system as a mapping from $(x_n, y_c, z_n) \rightarrow (x_{n+1}, y_c, z_{n+1})$ each time the phase space trajectory pierces the surface of section in the same direction. For the KSS circuit, constructing a return map using a sequence of maxima of x corresponds to selecting the $y \equiv \dot{x} = 0$ plane as the surface of section. The existence of the 1D curve on the surface of section implies that there is a function of the form $z_n = h(x_n, p_{n-1})$. The subscript on p indicates that the value of p may vary each time the map is iterated, because this parameter will be perturbed during chaos control.

We define $p_n \equiv p_0 + \delta p_n$ as the value of the parameter as $x_n \rightarrow x_{n+1}$ and p_0 as the constant value for which there is uncontrolled chaos. Thus the $n-1$ subscript denotes the value of p as the dynamical variable evolves from x_{n-1} to x_n. This point is important: z_n is determined by x_n based on the position of the attractor in phase space due to the value of p at the end of the nth cycle, p_{n-1}, not the value of p at the beginning of the $n+1$ cycle, p_n. The mapping between successive points on the surface of section means that x_{n+1} is determined by a function of the form $x_{n+1} = g(x_n, z_n, p_n) = g(x_n, h(x_n, p_{n-1}), p_n)$, which can be written equivalently as

$$x_{n+1} = f(x_n, p_n, p_{n-1}). \tag{3}$$

Equation (3) is the 1D return map that results from the 1D curve on the surface of section. If the parameter p is constant during the evolution of the dynamics (as it is for the map shown in Fig. 3), the return map reduces to $x_{n+1} = f(x_n, p_0)$. But during the application of perturbations, x_{n+1} may depend on both p_n and p_{n-1}.

We expand the return map to first order about the fixed point x_F for some periodic orbit at the parameter value p_0:

$$\delta x_{n+1} \approx \mu \delta x_n + \nu \delta p_n + \omega \delta p_{n-1}, \tag{4}$$

where $\delta x_n = x_n - x_F$, $\mu = \partial f / \partial x_n$, $\nu = \partial f / \partial p_n$, and $\omega = \partial f / \partial p_{n-1}$ with all derivatives evaluated at x_F and p_0. For a chaotic system $|\mu| > 1$, which means the value of δx_n grows over successive iterations of the map and the fixed point is unstable.

To control chaos, the perturbation δp_n must force the system toward the fixed point. Also, for the control algorithm to be stable, δp_n cannot grow over successive iterations. One way these conditions can be strongly satisfied is by requiring that $\delta x_{n+2} = 0$ and $\delta p_{n+1} = 0$. The latter requirement moves the system to the fixed point in a single control step; further control perturbations are only needed to address the motion of the system away from the fixed point due to noise. Then starting with Eq. (4), iterating it a second time, and solving both equations simultaneously for δp_n yields $\delta p_n \approx K \delta x_n + R \delta p_{n-1}$, where

$$K = -\frac{\mu^2}{(\mu\nu + \omega)}, \tag{5a}$$

$$R = -\frac{\mu\omega}{(\mu\nu + \omega)}. \tag{5b}$$

For the recursive proportional feedback algorithm, this approximation is set to an exact equality:

$$\delta p_n = K \delta x_n + R \delta p_{n-1}. \tag{6}$$

The first term on the right-hand side of Eq. (6) is proportional to the difference between the current state of the system and the fixed point, and the second term depends recursively on the previous perturbation. Derivatives of the return map for uncontrolled chaos determine the coefficients K and R used in the control feedback loop. In the broader context of control theory, one may study the entire range of values of K and R for which the system will approach periodic behavior. This procedure is well described in Ref. 3. By using control theory one can show that the fastest approach to control is

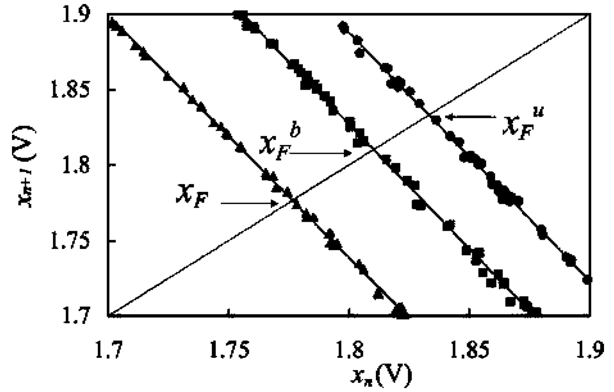

Fig. 4. The unperturbed (triangles), up (circles), and back (squares) maps near the $x_{n+1}=x_n$ line, generated experimentally according to the procedure in Sec. III with $\Delta p=0.05$ V. Linear fits to the data are also shown. The fixed points x_F, x_F^u, and x_F^b occur at the intersections of the maps and the $x_{n+1}=x_n$ line.

obtained by the choice of K and R in Eq. (5).

There is an effective experimental procedure for finding the slopes μ, ν, and ω that determine K and R. First we allow the system to run unperturbed at a constant parameter value p_0 for which the dynamics are chaotic. Then the 1D return map $x_{n+1}=f(x_n,p_0)$ is constructed and a linear fit to the map in the neighborhood of the $x_{n+1}=x_n$ line is made. The fixed point x_F is defined by the intersection of the map with the $x_{n+1}=x_n$ line,[26] and μ is the slope of the fit. To find ν and ω, it is necessary to find the dependence of the map on both p_n and p_{n-1}. This dependence is found by collecting data while repeatedly increasing the control parameter to $p_0+\Delta p$ for one oscillation and decreasing it to p_0 for the next oscillation, where Δp is a small, fixed value. Alternate pairs (x_n,x_{n+1}) of this sequence are on different return maps, designated the *up* and *back* maps. For the up map $p=p_0$ during the cycle that generates x_n and $p=p_0+\Delta p$ during the cycle that generates x_{n+1}, and conversely for the back map.[27] Fits to the up and back maps can be made and their respective fixed points, x_F^u and x_F^b, determined. Figure 4 shows the three return maps in the neighborhood of their fixed points for the KSS circuit. The shifts of the fixed points for the up and back return maps are a consequence of the changing location of the chaotic attractor in phase space as the control parameter is varied. In the neighborhood of their fixed points, each of the maps in Fig. 4 is approximately linear with slope μ:

$$f(x_n) = \mu x_n + (1-\mu)x_F, \tag{7a}$$

$$f^b(x_n) = \mu x_n + (1-\mu)x_F^b, \tag{7b}$$

$$f^u(x_n) = \mu x_n + (1-\mu)x_F^u. \tag{7c}$$

The derivatives ν and ω can be approximated by

$$\nu \approx \frac{f^u(x_F)-f(x_F)}{\Delta p} = (1-\mu)\frac{x_F^u - x_F}{\Delta p}, \tag{8a}$$

$$\omega \approx \frac{f^b(x_F)-f(x_F)}{\Delta p} = (1-\mu)\frac{x_F^b - x_F}{\Delta p}. \tag{8b}$$

Thus μ, ν, and ω, which determine the coefficients in the recursive proportional feedback algorithm, are found from the unperturbed, up, and back maps, which are constructed solely from experimental measurements of the output voltage x.

Note that if $x_F^b=x_F$, then $\omega=0$ and the recursive term in Eq. (6) goes to zero, which means that the back map falls on the unperturbed map when $p_n=p_0$, even though p has been changed during both the current and previous oscillations. In other words, the location of the chaotic attractor depends only on the current p_n and not on the previous value of p. In this case recursive proportional feedback reduces to the simple proportional feedback algorithm of Ref. 18:

$$\delta p_n = -\tilde{K}\delta x_n, \tag{9a}$$

$$\tilde{K} = -\frac{\mu}{\nu}. \tag{9b}$$

Thus the experimental procedure for finding μ, ν, and ω also provides a predictor for the likelihood that the simpler control algorithm is applicable and the recursive term in recursive proportional feedback is unnecessary.

Simple proportional feedback effectively controls chaos in many systems whose dynamics exhibit a 1D return map. For example, we can use simple proportional feedback to stabilize chaotic oscillations that are solutions of the Lorenz equations.[28] See the Appendix for more details. However, simple proportional feedback cannot control chaotic dynamics in some systems even though the dynamics are highly dissipative (that is, reducible to a 1D mapping). In particular, using simple proportional feedback as a control strategy with $\tilde{K}$ set according to Eq. (9b), we failed to control chaotic oscillations in the KSS circuit for the parameters of our experiment. As Fig. 4 clearly shows, the back map does not coincide with the unperturbed map for the KSS circuit, which explains why simple proportional feedback with $\tilde{K}=-\mu/\nu$ is not likely to achieve control for this system. If we follow the more general formulation of Ref. 3, we can show that the speed with which a system with simple proportional feedback and $\tilde{K}$ specified by Eq. (9b) approaches control increases as $|\omega\tilde{K}|\to 0$. We can also show that simple proportional feedback with $\tilde{K}=-\mu/\nu$ must fail for $|\omega\tilde{K}|\geq 1$.

IV. RECURSIVE PROPORTIONAL FEEDBACK APPLIED TO A SIMPLE ELECTRONIC OSCILLATOR

To apply the recursive proportional feedback algorithm to the KSS circuit, we must acquire the output voltage x and supply the perturbed control voltage p after each oscillation. To do so, we can choose from a wide variety of programable data acquisition systems. For our experiment, we created programs in LabVIEW[29] to control a National Instruments data acquisition board that interfaced with the circuit. We acquired data at 50 Hz, a frequency that is sufficiently high to resolve the approximately 1 Hz signal and low enough to allow sufficient time between data points to implement the recursive proportional feedback algorithm. The precision of the measured signal was 0.3 mV. After each data point, the program determined whether a local maximum of x had been acquired. A local maximum is identified when the previous

Table I. Calculated values of the quantities used to implement the recursive proportional feedback algorithm for the KSS circuit from experimental measurements of the output voltage x.

x_F	1.777±0.007 V
x_F^u	1.833±0.010 V
x_F^b	1.810±0.008 V
μ	−1.604±0.010
ν	2.92±0.64
ω	1.72±0.55
K	0.87±0.34
R	−0.93±0.57

voltage value is larger than both its previous and the current value. This simple test never returned false peaks because the noise in the circuit was sufficiently small. When control was turned on, the program used the recursive proportional feedback algorithm, Eq. (6), to calculate the voltage perturbation δp and update the value of $p=p_0+\delta p$ that was input to the circuit via the digital-to-analog output of the board. Because the perturbations must be small enough so that the linear approximation Eq. (4) is valid, the perturbations were set to zero if $|\delta p_n|>\epsilon$. For our experiment, $p_0=1.0000$ V and $\epsilon=0.05$ V. The precision of the input voltage p was 0.3 mV, which is an order of magnitude smaller than values of the perturbations δp. From the experimentally generated return maps shown in Fig. 4, we calculated the values of quantities needed for the recursive proportional feedback algorithm, which are shown in Table I. The uncertainties in these quantities are propagated from the uncertainties in least squares fits to the return maps.

The effect of the recursive proportional feedback algorithm on the signal is dramatically evident in Fig. 5, which shows the oscillating output voltage $x(t)$ versus time. Perturbations to the control parameter p began slightly before $t=20$ s. Before control is turned on, the oscillations vary aperiodically with a wide range in amplitude. After control is turned on, the voltage oscillates periodically with nearly constant amplitude. Figure 6 shows a sequence of the maxima x_n of the output voltage for an experimental run in which control was turned on and then turned off. Before control, the values of the maxima vary widely with a range of about 0 to 2.6 V. After control is turned on at $n=140$, the maxima have a nearly constant value very close to the target fixed point. The average of the actual maxima is $\bar{x}_n=1.778$ V, which is within the uncertainty of $x_F=1.777\pm0.007$ V. The standard deviation in the measured values of the maxima during control equals 0.002 V and is only 0.25% of the standard deviation of the maxima without control. When control is turned off at $n=900$, the maxima again vary widely. Figure 6 also shows the values of perturbations δp applied to the control parameter. Control was achieved immediately after the perturbations were generated and was lost once the perturbations were stopped. Control is achieved with remarkably small perturbations. The average absolute value of δp is only 1.4 mV, which means that the input current to the circuit during control is only increased or decreased on average by 0.14% of the input current with no control.

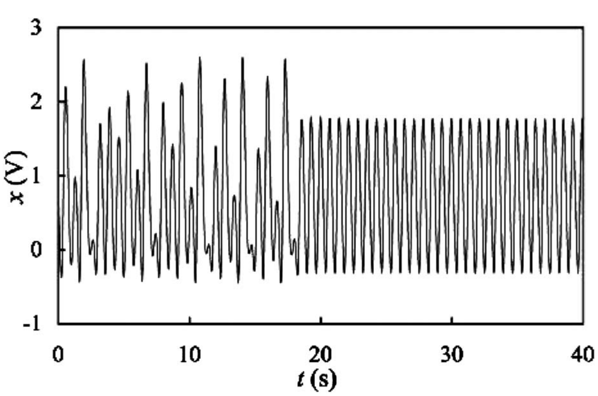

Fig. 5. Oscillating output voltage $x(t)$ of the circuit versus time. Control was turned on slightly before 20 s. For visualization purposes, the data was sampled at 1 kHz.

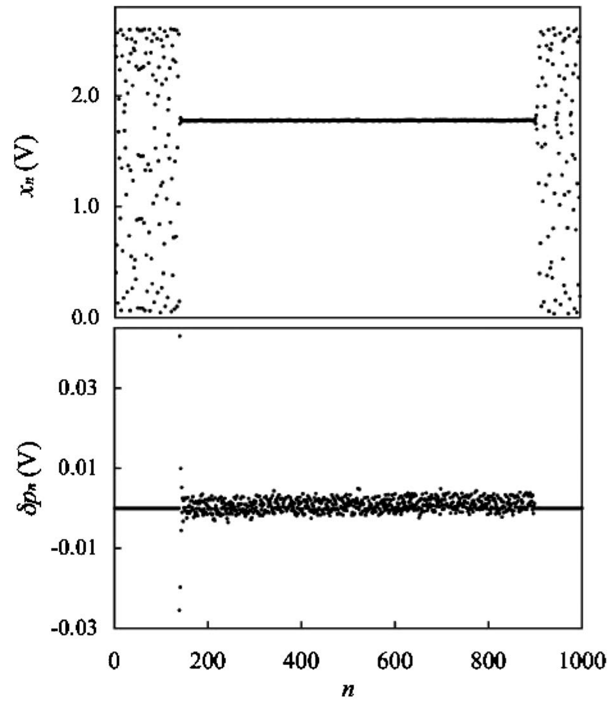

Fig. 6. A sequence of oscillation maxima x_n for an experimental run for which control was turned on at $n=140$ and turned off at $n=900$. The corresponding sequence of control perturbations δp_n is also shown.

V. CONCLUSIONS

We have derived the recursive proportional feedback algorithm and shown that it can be used to control chaotic oscillations in the Kiers, Schmidt, and Sprott electronic circuit. Control is achieved with small perturbations and the mean oscillation maximum during control is well within the uncertainty of the target fixed point. The values of the coefficients used in the recursive proportional feedback algorithm were calculated from experimentally measured values of the output voltage of the circuit during precontrol measurements. Recursive proportional feedback is suitable for highly dissipative systems, of which the KSS circuit is an example. Simple proportional feedback is also suitable for some highly dissipative systems, but cannot be used for the KSS circuit because the movement of the system's chaotic attractor through phase space depends on both the current and previous perturbations.

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APPENDIX: SUGGESTIONS FOR FURTHER STUDY

Problem 1. Write a program to control the logistic map using simple proportional feedback, Eq. (9). The logistic map is defined as

$$x_{n+1} = f(x_n, p) = px_n(1 - x_n). \tag{A1}$$

This map behaves chaotically for $3.57 < p \leq 4$, except for small windows of periodicity. Choose a value such as $p_0 = 3.9$ for the unperturbed value of the system parameter. The fixed point x_F and derivatives μ and ν in Eq. (9) can be determined analytically without recourse to numerical data (see Ref. 5 for details on the derivation of simple proportional feedback for the logistic map). Iterate the logistic map a few hundred times without control to insure steady state chaotic behavior and then turn on control by updating $p = p_0 + \delta p$ after each iteration. Investigate what happens when the control is turned off.

Problem 2. Write a program to control the Lorenz oscillator using recursive proportional feedback, Eq. (6). The equations for the Lorenz oscillator are given by[28]

$$\dot{x} = \sigma(y - x), \tag{A2a}$$

$$\dot{y} = rx - y - xz, \tag{A2b}$$

$$\dot{z} = xy - bz, \tag{A2c}$$

where σ, r, and $b > 0$ are system parameters. The dynamical variables x, y, and z behave chaotically for wide ranges of these parameters. Use r as the perturbation parameter to implement chaos control and choose $\sigma = 10$, $r_0 = 28$, and $b = \frac{8}{3}$. Numerically solve Eq. (A2) and treat the numerical solution for the dynamical variable z as experimental data. Find the sequence of maximum values of z. Follow the procedure in Sec. III for constructing unperturbed, up, and back return maps from this sequence and use these maps to find the fixed point z_F and derivatives μ, ν, and ω that determine K and R. If $\omega \approx 0$, try neglecting the recursive term in Eq. (6) and controlling the Lorenz oscillator using simple proportional feedback, Eq. (9).

Problem 3. Construct a chaotic oscillator using a nonlinear subcircuit $D(x)$ other than the one used in the KSS circuit. See Ref. 23 for several options for $D(x)$. For example, build a circuit with $D(x) = |x|$. Explore the chaotic dynamics of this circuit and compare the experimental output to numerical solutions of Eq. (2). Try to control chaos in this circuit using recursive proportional feedback.

Problem 4. Follow the control theory formulation of Ref. 3 and show that the linearized evolution of a highly dissipative system, subject to a 1D return map, may be expressed by the matrix equation:

$$\begin{pmatrix} x_{n+1} - x_F \\ p_n \end{pmatrix} = \begin{pmatrix} \mu + \nu K & \omega + \nu R \\ K & R \end{pmatrix} \begin{pmatrix} x_n - x_F \\ p_{n-1} \end{pmatrix}. \tag{A3}$$

Derive the recursive proportional feedback expressions for K and R, Eq. (5), by requiring both of the eigenvalues of the transformation matrix to be zero. For the values of μ, ν, and ω used in this paper, find the range of values of K and R that will achieve control (that is, those for which the eigenvalues of the transformation matrix have absolute values less than 1).

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[26] An interesting feature of the return map for the KSS circuit is that the thin 1D curve bends back near itself, creating closely separated lower and upper branches (see Fig. 9 in Ref. 22), which are not resolved by the scale used in Fig. 3. Thus there are two intersections of the map and the $x_{n+1} = x_n$ line, which means there are two unstable period-one fixed points. We used recursive proportional feedback to stabilize the lower-branch fixed

point. We were not able to gain control for the upper-branch fixed point, which suggests that it is not a saddle point. See Ref. 1 for details on classifying fixed points and Ref. 16 for a discussion of why a saddle point is needed for proportional feedback control.

[27] If the sequence is generated with the odd values of x corresponding to $p=p_0$ and the even values corresponding to $p=p_0+\Delta p$, then the up map consists of x_2 plotted against x_1, x_4 plotted against x_3, and so on; the back map consists of x_3 plotted against x_2, x_5 plotted against x_4, and so on.

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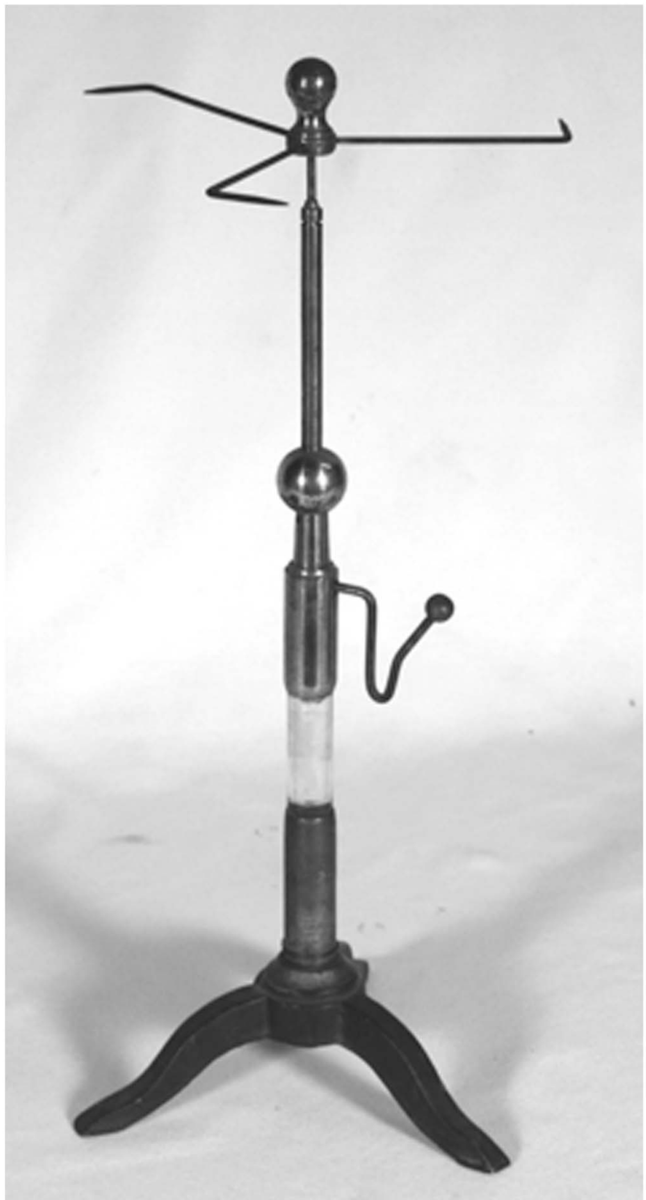

Electric Fly. The Electric Fly is made of a number of brass wires, with rearward-facing sharp points, joined at a hub. This is pivoted atop an insulated shaft. One terminal of an electrostatic machine is connected to the whirl and the other is grounded. As the charge builds up on the metallic parts of the whirl, the equipotential lines are bunched together at the sharp points, creating a large electric field there. Eventually the field becomes large enough to ionize the air molecules and create a space charge that is of the same sign as the point. The mutual repulsion between the space charge and the point causes the wheel to spin. This example, in the Greenslade Collection, runs well with either a Wimshurst machine or a Van de Graaff generator. (Photograph and Notes by Thomas B. Greenslade, Jr., Kenyon College)