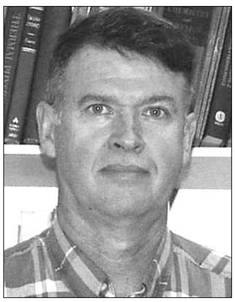
# Student-Friendly Precision Pendulum

*Randall D. Peters*



***Randall Peters** received his Ph.D. from the University of Tennessee in 1968 and is currently Chairman of the Department of Physics at Mercer University. In the last 12 years, he has studied a large number of mechanical oscillators, focusing on chaotic and complex behavior of the type to be found at http://physics.mercer.edu/petepag/nonlin.htm. The research described at this web page was motivated largely by his invention of the patented symmetric differential capacitive (SDC) sensor, which is the basis for several laboratory instruments marketed by TEL-Atomic, Inc., of Jackson, MI.*

***Department of Physics***
***Mercer University***
***1400 Coleman Ave.***
***Macon, GA 31207***

The pendulum has long been a favorite instrument for measuring Earth's gravitational field, i.e., $g \approx 9.8$ N/kg, or the free-fall acceleration, $g \approx 9.8$ m/s$^2$. However, it is not a trivial matter to measure $g$ to better than about 1% with a simple pendulum—one in which a concentrated mass swings at the end of a flexible cord. The biggest challenge in this case is the accurate determination of the distance from the support to the "center" of the pendulum.[1] To obtain accuracies in the vicinity of 1 part in $10^4$ has typically required a long pendulum. This has disadvantages. As the length of a pendulum increases, it is increasingly susceptible to noises, both from surrounding air and also from the support, which is never completely inert.[2]

Moreover, measurement with a naive pendulum will not show variations in $g$ from one place to another. For example, the altitude must increase from sea level to 3000 m (~10,000 ft) for $g$ to decrease by 1 part in a thousand. Similarly, the extreme sea-level global variation with latitude and/or longitude is only about 5 parts per thousand. Because gravity differences over Earth's surface are very small, geophysicists have used the milligal ($10^{-5}$ m/s$^2$) to state differences. In these units, going from the equator (where the mean value of $g$ = 978,049) to the poles, $g$ increases by 5172.[3] The reference for these gravimeter-based (relative) values is the absolute $g$ experiment that was performed with six Kater pendulums in Potsdam, Germany (latitude 52°N → 981,274) in the early part of this century.

With the information provided in this article, students should be able to easily measure the gravitational field strength to a few parts in 10,000, a degree of precision one to two orders of magnitude better than can be achieved with the usual simple pendulum under comparable conditions. The instrument involved is a "simple," slightly adjustable Kater pendulum.

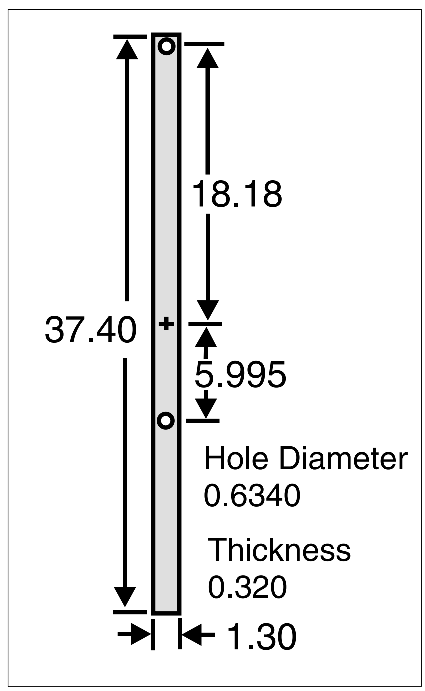



**Fig. 1. Design of student-friendly brass pendulum; measurements are in centimeters.**

## The Kater Pendulum

Many of the difficulties of the simple pendulum are removed by working with the reversible pendulum first considered by Captain Henry Kater in 1815. Being "reversible," the Kater pendulum oscillates about either of two axes. In conventional form, there is either an adjustable knife edge or a moving mass that is positioned, ideally, so that the period of the pendulum is the same about the two pivots. When the periods are matched so that $T_1 = T_2 = T$, it is easy to show that $T = 2\pi\sqrt{\ell/g}$ where $\ell$ is the spacing between the two axes.[4] Thus the system is equivalent to a simple pendulum of length $\ell$. Since $\ell$ can be measured with relative ease to a few parts in $10^4$, the Kater pendulum is an attractive means to accurately measure $g$.

For those not familiar with the Kater pendulum, a pedagogical example using a meterstick is provided in the Appendix. The theoretical estimated period obtained there ($T$ = 1.6 s) easily can be compared with a crude experiment. Simply hold the meterstick between the fingernails of thumb and middle finger and measure period with a stopwatch. First, hold the meterstick at the zero end, then reverse the stick and hold it at the 67-cm mark. Of course the large damping demonstrates that this "system" is not practical for a precise measurement of $g$. The same proves true when we try to provide axes of rotation by using sharpened pins that barely press into the stick. Using small circular rods inserted through the stick to serve as axes will also be found unsatisfactory.

## Present Instrument

The conventional Kater pendulum can be difficult to use, however, partly because the range of period adjustability is usually quite large. As we searched for a user-friendly design suitable for student labs, we turned to the computer. As a result, the basic unit for our pendulum is a piece of brass (37.4 cm long, 0.320 cm thick, and 1.30 cm wide). The overall length of 37.4 cm was chosen so that the period would be nominally 1 s for rotation about both axis 1 (an end) and axis 2 (a point ~ 0.67 x 37.4 cm away from axis 1). For a uniform rectangular physical pendulum of length $L$, having small width and thickness compared with $L$, $T = 2\pi\sqrt{\frac{2L}{3g}}$ (see Appendix).

Placement of the axes (holes) in a Kater pendulum requires a more complicated geometry than an "idealized" case. So, with the aid of the computer, using the parallel-axis theorem and recognizing the hole as having negative mass, we drilled two quarter-inch holes with their centers positioned 18.18 and 5.995 cm from the midpoint of the brass strip, as shown in Fig. 1.

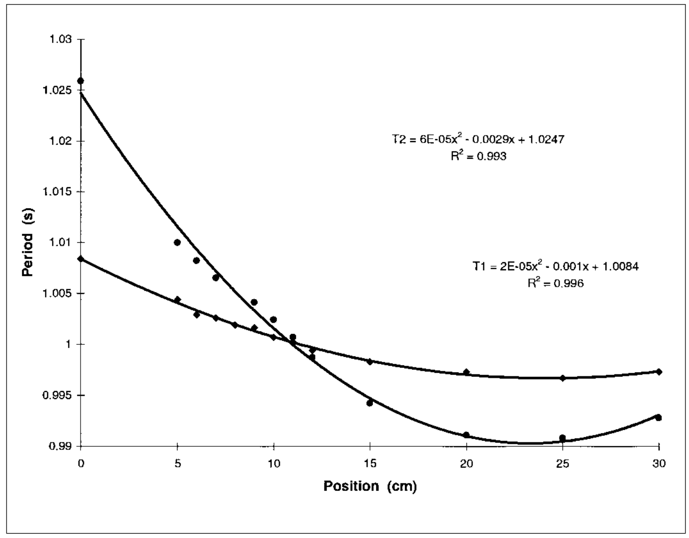



Fig. 2. Variation of period with binder-clip position.

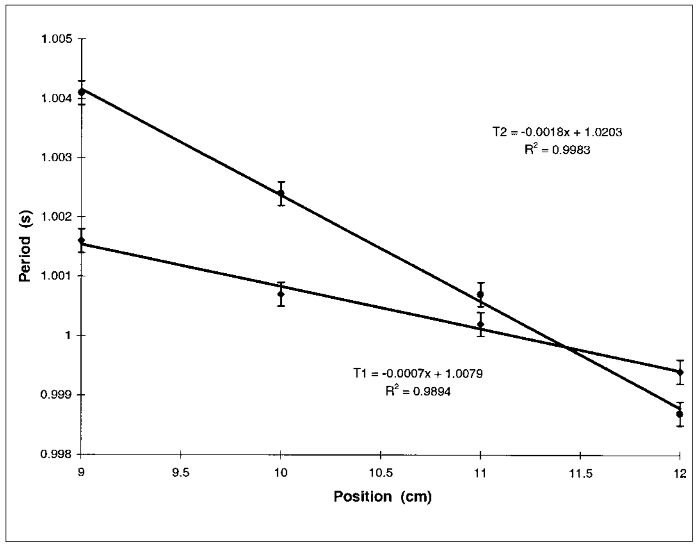



Fig. 3. Enlargement of Fig. 2 in region of period matching.

## The Knife Edge

A knife edge is required to hold the pendulum. A square cross section (0.25 x 0.25 in, ~ 1 in long) of carbon steel of the type used for cutting tools in a lathe was ground to an "edge," with the interior angle between planes roughly 30°. We clamped the unground end of the piece between the jaws of vice-grip pliers, which in turn were clamped to a conventional laboratory stand. For our brass pendulum, the log-decrement of the motion was small enough that its motion could still be visually perceived 30 min after initiation of motion. Note that the knife edge must have a rigid support; a flimsy one will both lengthen the period and increase the log-decrement.

## Period Adjustment

Typically, there are two ways to accomplish period adjustment—by moving the position of one of the pivots, or by changing the moment of inertia by altering the mass configuration. For a student-friendly apparatus, we chose the latter method.

To change the period around the nominal 1-s value, we used two ordinary metal spring clips of the type used to bind documents that are too thick for a paper clip. Each binder clip has a mass of 1.2 g, small compared with the pendulum mass of 129.3 g. By using a pair of clips, rather than just one, and positioning them on opposite sides of the pendulum, the center of mass of the system remains on the line of symmetry of the pendulum.

Operationally, students move the clips in increments of 1 cm, starting near one end of the pendulum and advancing toward the other end, measuring the period about each of the two axes for a given position of the clips.

## Timing

Period measurements were made using a standard photogate with the Precision Timer (Vernier software) system sold by PASCO, for their CI-6510 Signal Interface. The beam of the photogate was positioned vertically at the midpoint of whichever hole was not being used as an axis. The horizontal position was selected such that the beam was interrupted by the small segment of brass between one side of the pendulum and the near edge of the hole. For this placement of the photogate, the period measured will be other than ≈ 1 s if the amplitude is unacceptably large or small. Thus this arrangement proved useful in determining the onset of non-isochronism (amplitude large enough to increase the period through nonlinearity). A 30-min comparison of the timer system against WWV[5] showed that the times displayed by the timer were uniformly slightly long, so all data were corrected for this systematic error by dividing by 1.00038.

## Results

Data collected with the pendulum are shown in Figs. 2 and 3. Fitted to the data, which were plotted with Excel, are quadratic fits for which the $R^2$ values are close to Unity. For Fig. 3, we used a linear least-squares fit; the matched condition is one for which $T$ = 0.9999 s. The abscissa in both graphs corresponds to the clamping placement of the binder clips, with zero being near the end of the pendulum opposite axis 1. Position 36 (unused) corresponds to the clips being centered on axis 1 (top edge of the top hole).

Using the matched period value of 0.9999 s and the value of $\ell$ = 0.2481 m, the acceleration of gravity in Macon, Georgia, was measured to be $g = 4\pi^2\ell/T^2$ = 9.797 m/s$^2$.

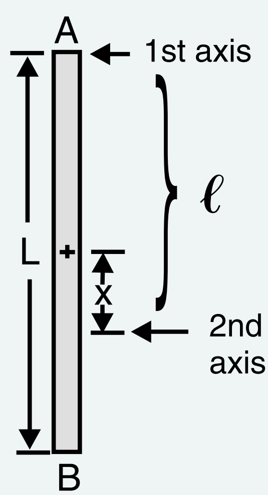
## Predicting the Uncertainty in *g*

Assuming random errors in the measurement of $\ell$ and $T$, the relative uncertainty in the acceleration of gravity is given by $\delta g/g = [(\delta\ell/\ell)^2 + 2(\delta T/T)^2]^{1/2}$. For our case, it was assumed that the machinist produced the pendulum (on a milling machine) with all dimensions to the nearest 0.001 in. Thus the relative uncertainty in $\ell$ is $\delta\ell/\ell = 1 \times 10^{-4}$. The random uncertainty in the period measurement (independent of the systematic error mentioned earlier, which was corrected) was determined by taking 100 measurements at a few points and calculating the standard deviation, which yielded $\delta T/T = 2 \times 10^{-4}$. With these numbers, the uncertainty in $g$ is determined to be 0.003 m/s$^2$.

## Leeways

We also constructed and evaluated a low-mass wooden (oak) pendulum, expecting that it would be a less precise instrument. It was fabricated "crudely" by cutting to the requisite rectangular shape on a table saw. The width turned out to be 1.25 cm, the thickness 0.42 cm, and the length 37.4 cm. The holes for the knife edge support were drilled with an ordinary ¼-in bit. Compared with the binder-clip mass total of 2.4 g, the mass of this pendulum was quite small at 13.2 g (an order of magnitude less than the 129-g brass pendulum). As expected, the timing errors proved larger, at approximately 3 ms—about an order of magnitude greater than those of the brass pendulum at 0.2 ms. Additionally, the length was not measured as precisely—the uncertainty being estimated at 0.2 mm. The variation of period about the two axes proved similar in trend to that of the brass pendulum, except that the range of variation was much greater, as expected. About axis 1, as

## *Appendix* – Meterstick "Kater" Pendulum




Consider a uniform stick of length $L$ = 1m, oscillating first about an axis at its upper end A, as shown in the figure. Our problem is to find a second axis, whose distance $x$ from the center toward the opposite end B, yields the same period as the first axis. Note that oscillation about this second axis requires that the meterstick be turned upside down.

We now calculate the periods of the pair, starting with the first axis. The moment of inertia of the uniform stick, with respect to its center, is $I_c = M L^2/12$ (assuming that both the width and the thickness are much smaller than the length of the stick). For rotation about the first axis at $L/2$ from the center, the parallel-axis theorem yields $I_1 = M\left(\frac{L}{2}\right)^2 + I_c = M L^2/3$. Thus, the period about the first axis is given by

$$T_1 = 2\pi\sqrt{\tfrac{1}{3}M L^2\Big/\left(\tfrac{1}{2}MgL\right)} = 2\pi\sqrt{\frac{2L}{3g}} = 2\pi\sqrt{\frac{2}{3g}} \approx 2\pi\sqrt{\frac{2}{30}} = 1.6 \text{ s}.$$

Now turn the meterstick upside down from the first position and allow it to swing about an arbitrary second axis at distance $x$ from the center toward B. Here the moment of inertia is given by $I_2 = M x^2 + I_c$, and the period is $T_2 = 2\pi\sqrt{\frac{x^2 + 1/12}{xg}}$. Setting $T_2 = T_1$ yields the quadratic equation $x^2 - \frac{2}{3}x + \frac{1}{12} = 0$, which has roots $x = 1/2$ (non-interesting) and $x = 1/6$, which is the Kater pendulum case. In particular, note that the distance between the two axes is $\ell = (1/2 + 1/6)L = 2L/3$.

Thus, the meterstick Kater pendulum is described by $T = T_1 = T_2 = 2\pi\sqrt{\frac{\ell}{g}} \approx 2\pi\sqrt{\frac{2/3}{10}} \approx 1.6$ s.

the clips were moved from 0 to 34 cm, the period varied through a total range of 87 ms, compared with 9 ms for the brass; likewise, about axis 2, the total range was 0.42 s, compared with 36 ms for the brass. With this wooden pendulum, the periods were found to match (from a linear least-squares fit) for the clips at 11.4 cm, yielding $T$ = 1.003 s, and an estimate for the acceleration of gravity, $g = (9.77 \pm 0.05)$ m/s$^2$. It was thus demonstrated that a crude, low-mass wooden Kater pendulum can do as well as a decent simple pendulum, although it is not recommended that such a pendulum be built.

Based on these results, we expect that a reasonably good pendulum might be made of soft aluminum, cut with shears. It is recommended, however, that a metal of higher density be used, and at the very least cut with a bandsaw and then filed to shape. Preferably, these operations should be performed on a milling machine. Not only will this yield an instrument whose dimensions are closer to the nominal values indicated in Fig. 1, but a separate measurement of the distance between the holes is then also unnecessary, assuming that the translating stages of the machine are properly calibrated.


### Acknowledgment

The author acknowledges the expert craftsmanship of Richard Hernandez of Texas Tech University, who built our student-friendly, brass precision pendulum.


### References


1. It is the center of percussion, and not the center of mass (c.m.) that determines the length of the pendulum; cf., C. Swartz, "The Plumb Line," *Phys. Teach.* **32**, 544 (1994). For a 1-s pendulum (spherical bob of 4-cm diameter), using the c.m. underestimates the period by one part in a thousand.
2. For an appreciation for the physics involved in the multiplicity of correction terms, see R. A. Nelson and M. G. Olsson, "The pendulum—rich physics from a simple system," *Am. J. Phys*. **54**, 112-121 (1986).
3. A. N. Strahler, *Plate Tectonics* (GeoBooks Publishing, Cambridge, MA, 1998), p. 88.
4. A. P. Arya, *Introduction to Classical Mechanics* (Allyn and Bacon, Needham Heights, MA, 1990), p. 342.
5. WWV are the call letters for the timing signals broadcast by the National Institute of Standards and Technology from Fort Collins, Colorado, at frequencies of 2.5, 5, 10, 15 and 20 MHz.