Coupled Pendulums: An Advanced Laboratory Experiment

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IN the study of oscillating systems many examples are found in which the interaction between the driving and the driven systems cannot be neglected. This is true whether driver and driven are two distinct systems or the driver is simply one mode of oscillation of a single system that is capable of vibrating in several different ways. Much insight into the fundamental physics of such systems can be secured through a quantitative experiment performed with very simple equipment.

Satisfactory apparatus consists of two small lead spheres attached to fish line strings so as to be supported as simple pendulums from a horizontal fish line, as shown in Fig. 1. It is convenient to pass one end of this string over a pulley so that the force in the string may be varied. It is easy to vary the length of one of the pendulums and also to alter their distance of separation on the horizontal string. This equipment is thus sufficiently flexible to allow the investigation of several aspects of the motion of coupled oscillators. It has been found that the coupling results from lateral motion of the supporting member. This is true even if a stiff wire is used for support and the pendulum strings or wires are rigidly attached to it. Such a situation is depicted in the "top view" of Fig. 1. The experiments discussed herein have been performed with a symmetrical system ($a=b$), and the angular amplitude of motion has been kept constant and small.

Theory of Coupled Oscillators

The theory of coupled oscillators is ably presented in several books. A very satisfactory treatment is given by Morse,[1] and the present outline is based on it.

The differential equations of motion of the two oscillators are, assuming equal masses,

$$\begin{aligned} m\,d^2x_1/dt^2 &= -K_1x_1+K_3x_2, \\ m\,d^2x_2/dt^2 &= -K_2x_2+K_3x_1, \end{aligned} \qquad (1)$$

where K_3 is the coupling coefficient, or force in dynes on oscillator *1* per unit displacement of oscillator *2* and *vice versa*.

Constants ν_1, ν_2 and μ are introduced as follows:

$$K_1=4\pi^2\nu_1^2m,$$
$$K_2=4\pi^2\nu_2^2m,$$
$$K_3=4\pi^2\mu^2m;$$

ν_1 and ν_2 are the free, uncoupled frequencies of vibration of oscillators *1* and *2*, respectively, and μ is a new coupling coefficient. The solutions of the differential equations of motion are of the form $x_1=f_1(t)$ and $x_2=f_2(t)$. These will in general be nonsinoidal and not even periodic. If it is demanded that the solution be simple harmonic, one finds that this condition can be satisfied provided the system is set in motion under proper conditions. For a pair of such coupled oscillators there are two ways in which the system will oscillate simple harmonically, and the frequencies ν_+ and ν_- are, respectively, higher than and lower than either ν_1 or ν_2; ν_+ and ν_- are the two roots of the equation

$$\nu=\{\tfrac{1}{2}(\nu_1^2+\nu_2^2)\pm\tfrac{1}{2}[(\nu_1^2-\nu_2^2)+4\mu^4]\}^{\frac{1}{2}}. \qquad (2)$$

These two modes of oscillation are called the *normal modes*. Their importance is due to the fact that the general motion can always be represented as a combination of the normal modes. When expressed in this way, the solutions of the equations of motion are

$$\begin{aligned} x_1 &= \frac{A_+\cos\alpha}{m^{\frac{1}{2}}}\cos(2\pi\nu_+t-\Phi_+) + \frac{A_-\sin\alpha}{m^{\frac{1}{2}}}\cos(2\pi\nu_-t-\Phi_-), \\ x_2 &= \frac{A_+\sin\alpha}{m^{\frac{1}{2}}}\cos(2\pi\nu_+t-\Phi_+) - \frac{A_-\cos\alpha}{m^{\frac{1}{2}}}\cos(2\pi\nu_-t-\Phi_-). \end{aligned} \qquad (3)$$

The four constants A_+, A_-, Φ_+ and Φ_- are arbitrary, and their values are fixed by specifying initial displacements and velocities of the two masses; α is defined by the relations

$$\tan\alpha=\frac{\nu_1^2-\nu_+^2}{\mu^2}=\frac{\mu^2}{\nu_2^2-\nu_+^2}=\frac{-\mu^2}{\nu_1^2-\nu_-^2}=-\frac{\nu_2^2-\nu_-^2}{\mu^2}. \qquad (4)$$

[1] P. M. Morse, *Vibration and sound* (McGraw-Hill, 1936), pp. 30–45.

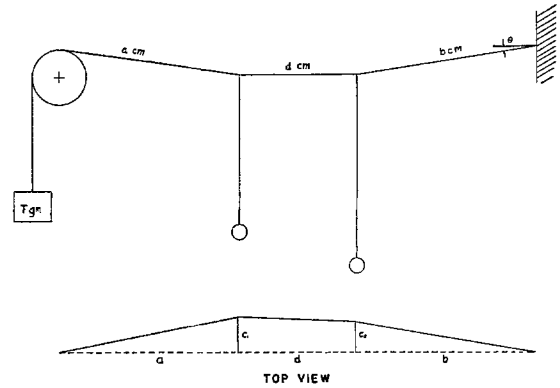

FIG. 1. Diagram of apparatus.

Two subdivisions can be introduced: (i) the case of resonance ($\nu_1=\nu_2$), and (ii) the case of distuning ($\nu_1\neq\nu_2$).

Resonance

The experiments dealing with resonance will be considered first. Equation (2) becomes

$$\nu=(\nu_1{}^2\pm\mu^2)^{\frac{1}{2}},$$

and, because μ is much smaller than ν_1 for the coupling encountered in this experiment, we can write,

$$\begin{aligned}\nu_+&=\nu_1+(\mu^2/2\nu_1),\\ \nu_-&=\nu_1-(\mu^2/2\nu_1),\\ \nu_+-\nu_-&=\mu^2/\nu_1.\end{aligned}\qquad(5)$$

For the case we are considering, the two simple pendulums will oscillate with frequency ν_+ if their initial displacements are equal in magnitude but opposite in direction. To produce the other normal mode of frequency, ν_-, the initial displacements should be equal in magnitude and in the same direction. If these frequencies are measured experimentally as a function of force in the supporting string for constant distance of separation, or *vice versa*, μ^2, and thus K_3, can be obtained as a function of these variables. A more accurate method of obtaining K_3 will be given presently, and this experiment may be performed primarily to direct attention to the fact that the two frequencies of the normal modes are always higher or lower than either of the natural frequencies of the pendulums and to give an indication as to the order of magnitude of K_3.

When a driver forces an oscillator at its natural frequency, theory predicts that displacement should lag behind the force by $\pi/2$ rad. This prediction can easily be verified by starting the pendulums in motion with the initial displacements $x_1=x_0$ and $x_2=0$ and the initial velocities equal to zero. For the first few seconds pendulum *1* is the driver, and the displacement of pendulum *2* is seen to lag by the expected $\pi/2$ rad.

Further study of the system put into motion under these initial conditions is very fruitful. The motion is not simple harmonic under these conditions but is a combination of the two normal modes and, as such, the displacements are given by Eqs. (3). These equations can be greatly simplified by applying the initial conditions and the resonance condition in addition to assuming small coupling. The resulting equations are

$$\begin{aligned}x_1&=\tfrac{1}{2}x_0(\cos 2\pi\nu_+t+\cos 2\pi\nu_-t),\\ x_2&=\tfrac{1}{2}x_0(\cos 2\pi\nu_-t-\cos 2\pi\nu_+t).\end{aligned}\qquad(6)$$

On substituting the equivalents of ν_+ and ν_- in terms of the resonance frequency ν_1, then expanding and simplifying, we have, finally,

$$\begin{aligned}x_1&=x_0\cos(\pi\mu^2t/\nu_1)\cos 2\pi\nu_1t,\\ x_2&=x_0\sin(\pi\mu^2t/\nu_1)\sin 2\pi\nu_1t.\end{aligned}\qquad(7)$$

It is thus seen that each oscillator has a vibration frequency ν_1 with an amplitude oscillation of frequency f, of magnitude $\mu^2/2\nu_1$, secured by equating $2\pi ft$ and $\pi\mu^2t/\nu_1$. The period of energy exchange, P, is the reciprocal of f and, as this is the quantity measured, we have $\mu^2=2\nu_1/P$. This leads directly to the coupling coefficient K_3.

Effect of tensile force on coupling.—To study the effect of tensile force on coupling it is convenient to use two pendulums of equal length, placed symmetrically on the supporting string ($a=b$ in Fig. 1, and $\nu_1=\nu_2$). Small lead spheres of mass about 100 gm serve as pendulum bobs. As long as the force applied to the end of the string passing over the pulley is quite large compared to the weight of the bobs, the angle θ between the string and a horizontal line at the fixed points (pulley and wall) will be quite small. Consideration of the three forces acting at the point of attachment of either pendulum to the supporting string leads directly to the conclusion that the product of $\tan\theta$ and the force T in the portion of the string between the pendulums is constant; or, if θ is small, $T\theta$ is constant. Now, the true length of the pendulums is $l+a\sin\theta$, or approximately $l+a\theta$. The lateral displacement of the point of attachment of pendulum to supporting string for either of the pendulums will thus be directly proportional to θ, and the coupling coefficient will therefore be proportional to θ. Combining this result with the foregoing relation for T and θ, we have $TK_3=C'$, a constant. It should be observed that, when θ is small, T is approximately equal to the force applied to the end of the string passing over the pulley.

Experimentally, the period of energy exchange, P, is determined as a function of the tensile force in the supporting member. Using a number of different tensile forces and securing K_3 for each from the observed periods of energy exchange by means of the formula $K_3=8\pi^2\nu_1 m/P$, one finds the dependence of K_3 on tensile force (Fig. 2). The data displayed in Fig. 2 were secured by a student. It is to be noted that this curve is in close agreement with the theoretically predicted relationship, $TK_3=C'$.

Effect of natural period on coupling.—Since $\mu^2=2\nu_1/P$ and $\nu_1=(1/2\pi)(g/l)^{\frac{1}{2}}$, we have $\mu^2=(1/\pi P)(g/l)^{\frac{1}{2}}$. Also, $\mu^2=k_3/4\pi^2 m$, which leads to $k_3=(4\pi m/P)(g/l)^{\frac{1}{2}}$. If the pendulums are kept in resonance and a series of observations is made on period of energy exchange as a function of the common lengths of the two pendulums, it is possible to verify the fact that the product of K_3 and $Pl^{\frac{1}{2}}$ is a constant. A typical set of data secured by a student is displayed in Fig. 3.

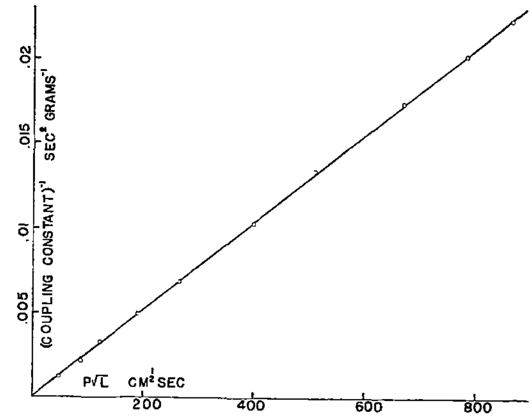

FIG. 3. Reciprocal of coupling coefficient K_3 *versus* $Pl^{\frac{1}{2}}$.

Effect of separation of the two pendulums on the coupling.—The closeness of coupling of the two resonant pendulums depends on their distance of separation. The manner of variation may be investigated by keeping the force in the supporting string constant, choosing a convenient length for the pendulums and measuring the period of energy exchange as a function of separation. The coupling coefficient K_3 is calculated from the period of energy exchange as in the previous experiments.

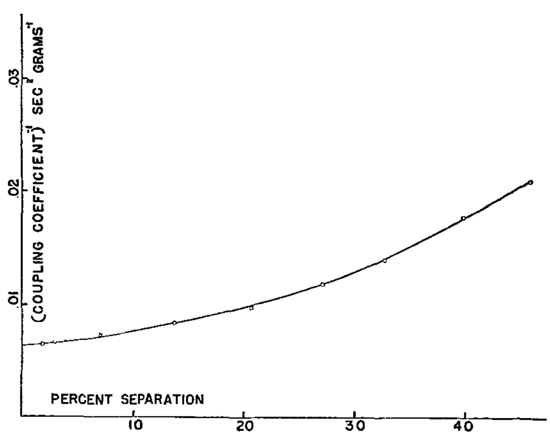

FIG. 4. Reciprocal of coupling coefficient K_3 *versus* percentage of separation.

As the percentage separation $100d/(2a+d)$ is increased, there is a reduction in the amount of lateral displacement of the point of attachment of either pendulum. The rate of decrease of this lateral displacement is less than the rate of increase of percentage separation. The coupling coefficient thus decreases slowly with increased separation of the pendulums. Superimposed on this is a practically linear decrease in coupling due to increased separation. (A given lateral motion of the point of attachment of one pendulum produces less and less coupling force on the second pendulum as their distance of separation increases.) The combination of these two effects accounts for the upward concavity of the curve of Fig. 4, which is a graph of the reciprocal of K_3 *versus* percentage separation. These results were also secured by a student.

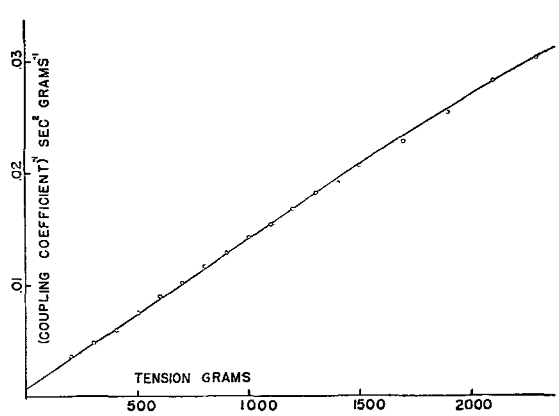

FIG. 2. Reciprocal of coupling coefficient K_3 *versus* tensile force.

Distuning

When the natural frequencies of the two pendulums are not equal, the amplitude of the

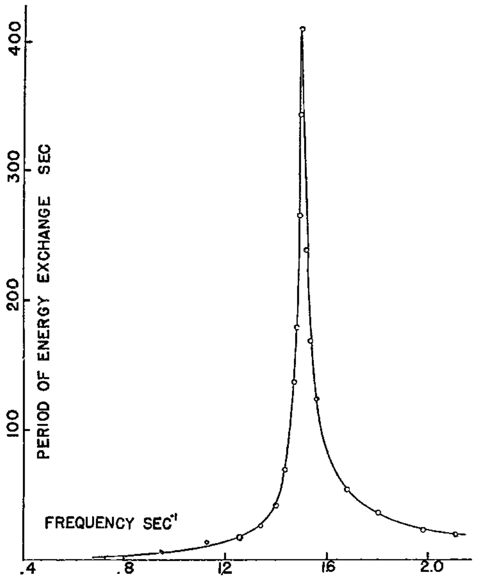

FIG. 5. Period of energy exchange *versus* frequency of the variable frequency pendulum.

pendulum which is originally displaced in order to start the system in motion will remain fairly constant, and will always be much larger than that of the second pendulum. While it cannot be said that this initially displaced pendulum exerts a simple harmonic force of constant amplitude on the other pendulum, many of the experimental results which are observed under such conditions are in approximate agreement with calculations obtained by considering the initially displaced pendulum to be a driver, and the other a driven oscillator. With the constant-length pendulum as the driver, the amplitude of the variable-length pendulum will be small, provided coupling is small and provided the variable-length pendulum is not in resonance.

If the pendulums are started in motion in a normal mode, then the amplitude ratio between them will be maintained and the steady-state amplitude of the driven oscillator can be calculated from the theory of forced oscillators.

If the distuned system is started according to the initial conditions,

$$x_1=x_0,\quad x_2=0,\quad (\mathrm{d}x_1/\mathrm{d}t)_0=(\mathrm{d}x_2/\mathrm{d}t)_0=0,$$

the motion will be nonperiodic, but rather simple solutions for x_1 and x_2 can be obtained from the general solutions of Eq. (3):

$$\begin{aligned} x_1&=x_0(\cos^2\alpha\cos 2\pi\nu_+t+\sin^2\alpha\cos 2\pi\nu_-t),\\ x_2&=x_0\sin\alpha\cos\alpha(\cos 2\pi\nu_+t-\cos 2\pi\nu_-t). \end{aligned}\tag{8}$$

Since $\tan\alpha[=-\mu^2/(\nu_1^2-\nu_-^2)]$ is small for small coupling, we see that $\cos\alpha$ is approximately unity and $\sin\alpha$ is approximately $-\mu^2/(\nu_1^2-\nu_2^2)$, ν_+ being just a little larger than ν_1, and ν_- just a little smaller than ν_2. Therefore,

$$\begin{aligned} x_1&\cong x_0\left[\cos 2\pi\nu_+t+\frac{\mu^4}{(\nu_1^2-\nu_2^2)^2}\cos 2\pi\nu_-t\right],\\ x_2&\cong\frac{2\mu^2x_0}{\nu_1^2-\nu_2^2}[\sin 2\pi\tfrac{1}{2}(\nu_++\nu_-)t]\\ &\qquad\times[\sin 2\pi\tfrac{1}{2}(\nu_+-\nu_-)t]. \end{aligned}\tag{9}$$

It is apparent from examination of these equations that the amplitude of pendulum *1* will undergo small variations while pendulum *2* will have a small amplitude which is modulated with a frequency $\frac{1}{2}(\nu_1-\nu_2)$.

As the length of the variable-frequency pendulum approaches that of the driver, its amplitude of motion increases and the period of energy exchange increases. If the period of energy exchange is plotted as a function of the frequency of the variable pendulum, a typical resonance curve results. If the dissipative forces are small, the resonance curve is sharp. Typical results secured by students are shown in Fig. 5. Similar curves may be secured by plotting amplitude of response as a function of the variable frequency.

The amplitude of the variable-frequency pendulum is, according to Eqs. (9), $A=2\mu^2x_0/(\nu_1^2-\nu_2^2)$. Within experimental error, measurements of A at various frequencies lead to a constant value for μ when substituted in this equation.

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This experiment has been used during the past two years in connection with a laboratory course in mechanics for junior students majoring in physics. The results secured by all the students who have performed the experiment have been uniformly good, and in all cases the experiment has stimulated a great deal of interest in the physics of coupled oscillators.

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