3 Dynamical models II.

Why is there a relationship between harmonic motion and uniform circular motion?

3.1 States of the system (2.2.2).

Jargon alert: bear with me as I develop some formal mathematical terminology that is useful in this context. Don't worry about memorizing the terms; we will just need to develop a precise language so we can describe some useful ideas. My notes will be more detailed than usual here.
Reference: Abraham & Shaw, *Dynamics: the Geometry of Behavior.* (handout)

- Identify appropriate coordinates and use these to map physical states onto points in a mathematical space, that we call *phase space* or *state space*, e.g. $\mathbb{R}^2 = \{x, \dot{x}\}$ for the position and velocity of a single object in one dimension, $\mathbb{R}^6 = \{x, y, z, \dot{x}, \dot{y}, \dot{z}\}$ for position/velocity in three dimensions. For a rigid body in three dimensions we need 12 dimensions, because we must also include rotational coordinates and velocities. For n bodies we need $12n$ dimensions, etc. (Aren't you glad we will be sticking to one-dimensional motion in this course??!) Sometimes it is useful to include an additional t-axis, and this is called *extended phase space*.

- Physical behavior can be represented mathematically as curves in extended phase space: $\{t, x(t), \dot{x}(t)\}$ in our $1D$ case (we will not try to draw curves in the $7D$ extended phase space for $3D$ particle motion!). We expect these to be continuous and smooth (differentiable). When you plot $x(t)$ and $\dot{x}(t)$ separately, you are plotting projections of a curve in $\{t, x, \dot{x}\}$ onto the $\{t, x\}$ and $t, \dot{x}$ planes.

3.2 Solve the DE (2.2.4).

Graphical method: the DE specifies a *vector field* on the phase space. Solutions to the DE are curves in phase space that are always tangent to the vector field.

- Introductory example: exponential growth and decay, $\dot{x} = rx$. Set $r = 1$; afterwards can rescale x axis to account for $r \neq$

1. Draw slope field in extended phase space, show that it is tangent to exponential curves. Note symmetry under $t \to t+a$ (autonomous equation).

- Harmonic oscillator equation, $\ddot{x} = -\omega_0^2 x$. Set $\omega_0 = 1$ for simplicity.
  - Independent of time.
  - Draw vector field projected along $\{x, \dot{x}\}$. See this *Java applet* on phase portraits of autonomous systems.
  - Phase curves are *circles*. Circulation in counter-clockwise direction.
  - Letting $x_1 = x, x_2 = \dot{x},$, we have

$$\frac{dx_2}{dx_1} = \frac{(dx_2/dt)}{(dx_1/dt)}.$$

  - The vector field can be determined from the phase space coordinate using a *linear transformation* that can be expressed in the following matrix form:

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x}/\omega_0 \end{bmatrix} = -\omega_0 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x}/\omega_0 \end{bmatrix}$$

    References: www.sosmath.com

    Shankar, *Basic Training in Mathematics*.

    Online matrix tutorial

  - Alternatively, we may combine the pair of real numbers $\{x, \dot{x}/\omega_0\}$ into a *complex* number $z \equiv x + i\dot{x}/\omega_0$. In this case the linear transformation can be expressed as

$$\frac{dz}{dt} = -i\omega_0 z.$$

    References:

    Shankar, *Basic Training in Mathematics* (handout).

    Dave's Short Course on Complex Numbers

    John and Betty's Journey into Complex Numbers (A short introduction in the form of a children's book.)