23 Fourier series.

23.1 Solving the wave equation.

d'Alembert's solution to the wave equation,

$$\psi(z,t) = f(z-ct) + g(z+ct),$$

involves two arbitrary functions (subject to physically reasonable assumptions concerning continuity and differentiability). It is a general solution to the wave equation—what more do we need?

23.1.1 Harmonic solutions.

In the harmonic oscillator problem, solutions of the form $x(t) = x_0 e^{i\omega t}$ allow us to reduce the differential equation $\ddot{x} + \omega_0^2 x = 0$ to an algebraic equation $-\omega^2 + \omega_0^2 = 0$. The same applies to the wave equation: with $\psi(z,t) = \psi_0 e^{i(\omega t \pm kz)}$,

$$\frac{\partial^2 \psi}{\partial t^2} - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial z^2} = 0 \quad \Rightarrow \quad -\omega^2 + \frac{k^2}{c^2} = 0.$$

The *dispersion relation* $\omega^2 = c^2 k^2$ tells us that harmonic time dependence $\psi(z_0,t) = \psi_0 e^{i\omega t}$ at some z_0 must be associated with harmonic spatial dependence with $k = \pm\omega/c$. We expect complex exponential solutions for the wave equation because it is linear and independent of the origin chosen for z, t; this is analogous to ordinary differential equations that are linear and autonomous (independent of the t origin).

23.1.2 How do we specify a function?

- A function $f : \mathbb{R} \to \mathbb{R}$ is a pairing of numbers:

$$\begin{aligned} f(0) &= 1, \\ f(0.1) &= 0.995004165\ldots, \\ f(0.111\ldots) &= 0.993833509\ldots, \\ f(\pi/2) &= 0, \\ &\text{etc.} \end{aligned}$$

- We may also describe f in terms of other known functions: a Taylor series, for example, uses polynomials.

23.2 Definition of the Fourier series.

A *Fourier series* expresses a periodic function $f(x+\lambda) = f(x)$ as a superposition of harmonic functions:

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(mkx) + \sum_{m=1}^{\infty} b_m \sin(mkx), \tag{23.1}$$

where $k = 2\pi/\lambda$.

23.2.1 Coefficients.

The coefficients $\{a_m, b_m\}$ are determined from $f(x)$ through the following integrals:

$$a_m = \frac{2}{\lambda}\int_{-\lambda/2}^{\lambda/2} f(x)\cos(mkx)\,dx, \tag{23.2a}$$

$$b_m = \frac{2}{\lambda}\int_{-\lambda/2}^{\lambda/2} f(x)\sin(mkx)\,dx. \tag{23.2b}$$

23.2.2 Complex exponential form.

We may also write

$$f(x) = \sum_{m=-\infty}^{\infty} c_m e^{imkx}, \tag{23.3}$$

$$c_m = \frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2} f(x) e^{-imkx}\,dx. \tag{23.4}$$

- If $f(x) \in \mathbb{R}$, then $c_{-m} = c_m^*$.
- The coefficients in Eqs. 23.1 and 23.3 are related by

$$c_m = \begin{cases} (a_m - ib_m)/2, & m > 0 \\ a_0/2, & m = 0 \\ (a_m + ib_m)/2, & m < 0 \end{cases}$$

- Note that $c_0 = a_0/2 = (1/\lambda)\int_{-\lambda/2}^{\lambda/2} f(x)\,dx$ is the average of $f(x)$ over one period.