

CHAPTER 1 PROBLEMS and ANSWERS

Problem (1.1).

Two charges, each $q=+1.6 \times 10^{-19}$ Coulombs, are located at $(0,0,a)$ and at $(0,0,-a)$ where $a=1.0 \times 10^{-9}$ meters.

(a) Calculate the electric field at the origin $(0,0,0)$.

(Answ: the field is zero.)

(b) Calculate the electric field at (a,a,a) .

(Answ: $\mathbf{E} = (6.07, 6.07, 1.96) \times 10^8$ Volts/m.)

(c) An electron, $q=-1.6 \times 10^{-19}$ Coulombs, flies through the point (a,a,a) with the velocity $\mathbf{v} = v_0(1,2,3)$ where $v_0 = 10^5$ m/sec. What forces are exerted on the electron due to the two stationary charges?

(Answ: $\mathbf{F} = q\mathbf{E} = (-9.71, -9.71, -3.14) \times 10^{-11}$

Newtons. There is no magnetic force.)

Problem (1.2).

At a certain moment a moving proton, $q=+1.6 \times 10^{-19}$ Coulombs, is located at $(0,0,a)$ with velocity components $v_0(1,1,0)$ where $a=10^{-9}$ m. and $v_0=10^5$ m/sec. At the same moment a moving electron, $q=-1.6 \times 10^{-19}$ Coulombs, is located at (a,a,a) with velocity components $(0,10^6,0)$ m/sec.

(a) Calculate the electric and magnetic fields at the position of the electron due to the proton.

(Answ: $\mathbf{E} = (E_0, E_0, 0)$ where $E_0 = 5.09 \times 10^8$ V/m.
and $\mathbf{B} = (0, 0, 0)$ because $\mathbf{v}_p \times \mathbf{E} = 0$.)

(b) Calculate the force on the electron due to the electric field of the proton.

(Answ: $\mathbf{F} = (-F_0, -F_0, 0)$ where $F_0 = |q|E_0 = 8.14 \times 10^{-11}$ N.)

(c) Calculate the force on the electron due to the magnetic field of the proton.

(Answ: $\mathbf{F} = \mathbf{v}_{\text{electron}} \times \mathbf{B} = 0$ N.)

(d) Calculate the electric and magnetic forces on the proton due to the fields generated by the electron.

Answ: The electric field at the position of the proton, $\mathbf{R} = (0, 0, a)$, due to the electron at $\mathbf{r} = (a, a, a)$ is given by

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{(-1.6 \times 10^{-19})}{r^3} \mathbf{r},$$

where $\mathbf{r} = \mathbf{R} - \mathbf{r} = (-a, -a, 0) = -a(1, 1, 0)$, where $a = 10^{-9}$ m.

Therefore

$$\mathbf{E} = (5.09 \times 10^8)(1, 1, 0).$$

The magnetic field at the position of the proton due to the motion of the electron is given by $c^2 \mathbf{B} = \mathbf{v} \times \mathbf{E}$, where the velocity of the electron is $\mathbf{v} = 10^6(0, 1, 0)$ m/sec.

$c^2 \mathbf{B} = (5.09 \times 10^{14})(0, 0, -1)$ so $\mathbf{B} = (0.566 \times 10^{-2})(0, 0, -1)$ Teslas.

The force on the proton due to the electric field is

$$\mathbf{F}_E = 8.15 \times 10^{-11} (1, 1, 0) \text{ N.}$$

The force on the proton due to the magnetic field is

$$\mathbf{F}_M = q(\mathbf{v}_p \times \mathbf{B}) = 0.906 \times 10^{-16} (-1, 1, 0) \text{ N.}$$

Problem (1.3).

A particle having a velocity $\mathbf{V} = v_1 \mathbf{u}_x$ carries a charge q_1 C and is located at the origin. A second particle, charge q_2 , is located at $\mathbf{r} = a \mathbf{u}_x + b \mathbf{u}_y + c \mathbf{u}_z$, and it has a velocity $\mathbf{V}_2 = v_2 \mathbf{u}_y$.

(a) Show that the force on charge #2 due to the magnetic field generated by charge #1 is

$$\mathbf{F}_{21} = \frac{\mu_0}{4} \frac{q_1 q_2}{r^3} b v_1 v_2 \mathbf{u}_x.$$

(b) Show that the force on charge #1 due to the magnetic field generated by charge #2 is

$$\mathbf{F}_{12} = - \frac{\mu_0}{4} \frac{q_1 q_2}{r^3} a v_1 v_2 \mathbf{u}_y. \text{ Notice that } \mathbf{F}_{21} \text{ does not equal } -\mathbf{F}_{12}$$

so that Newton's law of the equality of forces of action and reaction is not obeyed in this case.

Answer (1.3).

(a) The electric field at the position of particle #2 due to particle #1 is

$$\mathbf{E}_{21} = \frac{1}{4\pi\epsilon_0} \frac{q_1}{r^3} (a, b, c).$$

The magnetic field at the position of particle #2 due to the motion of particle #1 is given by

$$c^2 \mathbf{B}_{21} = \mathbf{v}_1 \times \mathbf{E}_{21} = \frac{1}{4} \frac{q_1 v_1}{r^3} (0, -c, b),$$

or

$$\mathbf{B}_{21} = \frac{\mu_0}{4} \frac{q_1 v_1}{r^3} (0, -c, b).$$

The magnetic force on particle #2 due to its motion is

$$\mathbf{F}_{2M} = q_2 (\mathbf{v}_2 \times \mathbf{B}_{21}) = \frac{\mu_0}{4} \frac{q_1 q_2 v_1 v_2}{r^3} (b, 0, 0).$$

(b) The electric field at the position of particle #1 due to particle #2 is

$$\mathbf{E}_{12} = - \frac{1}{4} \frac{q_2}{r^3} (a, b, c).$$

The magnetic field at the position of particle #1 due to the motion of particle #2 is given by

$$c^2 \mathbf{B}_{12} = \mathbf{v}_2 \times \mathbf{E}_{12} = - \frac{1}{4} \frac{q_2 v_2}{r^3} (c, 0, -a),$$

or

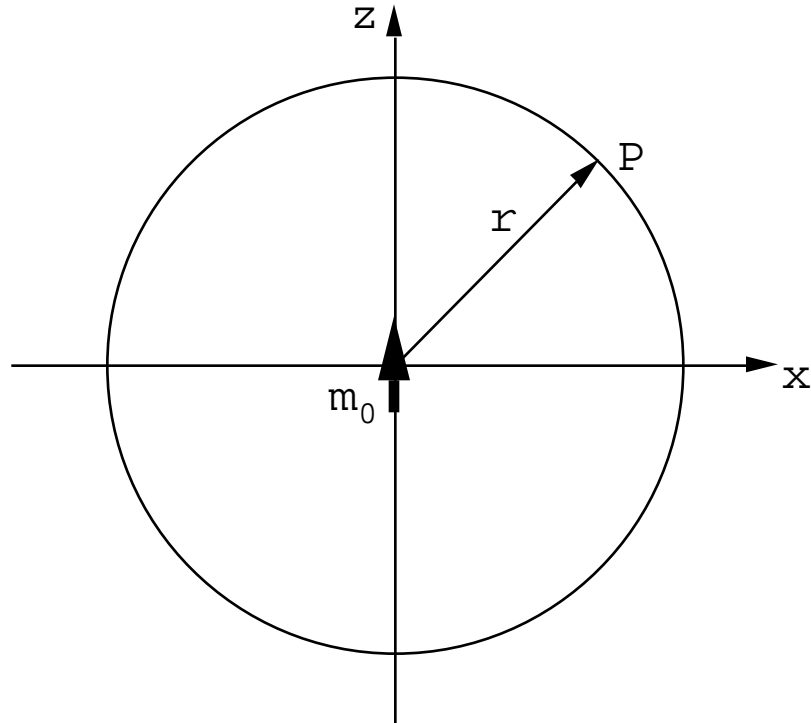
$$\mathbf{B}_{12} = \frac{\mu_0}{4} \frac{q_2 v_2}{r^3} (c, 0, -a).$$

The magnetic force on particle #1 due to its motion is

$$\mathbf{F}_{1M} = q_1 (\mathbf{v}_1 \times \mathbf{B}_{12}) = - \frac{\mu_0}{4} \frac{q_1 q_2 v_1 v_2}{r^3} (0, a, 0).$$

Problem (1.4).

An electron carries a magnetic moment of $|\mathbf{m}_0| = 9.27 \times 10^{-24}$ Joules/Tesla = 1 Bohr magneton. Suppose that this magnetic moment is oriented along the z-axis as shown in the figure.



(a) At what angle is the field measured by an observer at P a maximum?

(Answ: $\theta = \pm \pi/2$.)

(b) If $r = 1$ micron (10^{-6} m.) what is the magnitude and direction of this maximum field?

(Answ: $|\mathbf{B}_{\max}| = 18.54 \times 10^{-13}$ Teslas directed along +z).

(c) What is the minimum magnetic field? At what angle does it occur, and what is the direction of the field?

(Answ: $|\mathbf{B}_{\min}| = 9.27 \times 10^{-13}$ Teslas directed along $-z$.)

The observer is at $\theta = 0$ or π .)

Answer :

$$\mathbf{B} = \frac{\mu_0}{4\pi} \left[3 \frac{m_0 z \mathbf{r}}{r^5} - \frac{m_0 \mathbf{u}_z}{r^3} \right] \text{ therefore}$$

$$B_x = \frac{\mu_0 m_0}{4\pi} \frac{3xz}{r^5}, \quad B_y = \frac{\mu_0 m_0}{4\pi} \frac{3yz}{r^5}, \quad B_z = \frac{\mu_0 m_0}{4\pi} \left[\frac{3z^2}{r^5} - \frac{1}{r^3} \right].$$

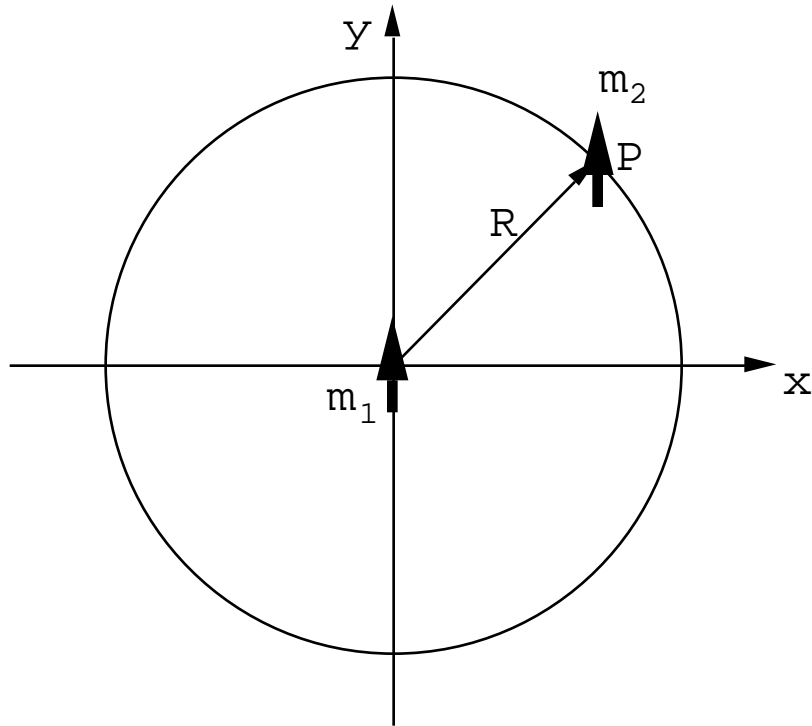
$$B^2 = \frac{\mu_0 m_0}{4\pi r^3} \left(1 + \frac{3z^2}{r^2} \right) \text{ so } B^2 \text{ is a maximum at } x=0, y=0, z=r.$$

$$B^2 \text{ is a minimum at } z=0. \quad B_{\min} = \frac{\mu_0 m_0}{4\pi r^3}. \quad B_{\max} = 2B_{\min}.$$

Problem (1.5).

The energy of interaction between two magnetic dipoles is given by $-\mathbf{m}_1 \cdot \mathbf{B}_2$ or by $-\mathbf{B}_1 \cdot \mathbf{m}_2$ where \mathbf{B}_1 is the field generated at the position of dipole #2 by dipole #1, and \mathbf{B}_2 is the field at dipole #1 generated by dipole #2. Let these two magnetic dipoles be separated by a constant distance $R = 10^{-6} \text{ m}$ ($1 \mu\text{m}$).

(a) Assume that the two dipoles are forced to remain parallel as shown in the figure. At what angle is the interaction energy a minimum? What is this minimum energy?



(Answ: $\theta = \pm \pi/2$, $U_{\min} = -2 \frac{\mu_0 m_1 m_2}{4 R^3}$.)

(b) Assume that $\theta = 0$ in the figure, but that the two dipoles are free to rotate in the x-y plane. Let $m_{1|x} = m_1 \cos \theta_1$ and $m_{1|y} = m_1 \sin \theta_1$. Similarly let $m_{2|x} = m_2 \cos \theta_2$ and $m_{2|y} = m_2 \sin \theta_2$. What will be the minimum energy configuration, and what will be the minimum energy?

(Answ: $\theta_1 = \theta_2 = 0$ or π . $U_{\min} = -2 \frac{\mu_0 m_1 m_2}{4 R^3}$.)

Answer:

(a) $\mathbf{B}_1 = \frac{\mu_0}{4} \frac{3(\mathbf{m}_1 \cdot \mathbf{r})\mathbf{r}}{r^5} - \frac{\mathbf{m}_1}{r^3}$, $x = R \cos \theta$, $y = R \sin \theta$, $z = 0$.

$$B_{1x} = \frac{\mu_0}{4} \frac{m_1}{R^3} 3\sin\theta \cos\theta, \quad B_{1y} = \frac{\mu_0}{4} \frac{m_1}{R^3} (3\sin^2\theta - 1),$$

therefore

$$U = - \mathbf{B}_1 \cdot \mathbf{m}_2 = - \frac{\mu_0}{4} \frac{m_1 m_2}{R^3} (3\sin^2\theta - 1).$$

This expression is clearly a minimum when $\sin\theta = \pm 1$.

(b) When $\theta = 0$ and $\mathbf{r} = (R, 0, 0)$ one finds

$$B_{1x} = \frac{\mu_0}{4} \frac{m_1}{R^3} 2 m_1 \cos\theta_1, \quad B_{1y} = - \frac{\mu_0}{4} \frac{m_1}{R^3} m_1 \sin\theta_1,$$

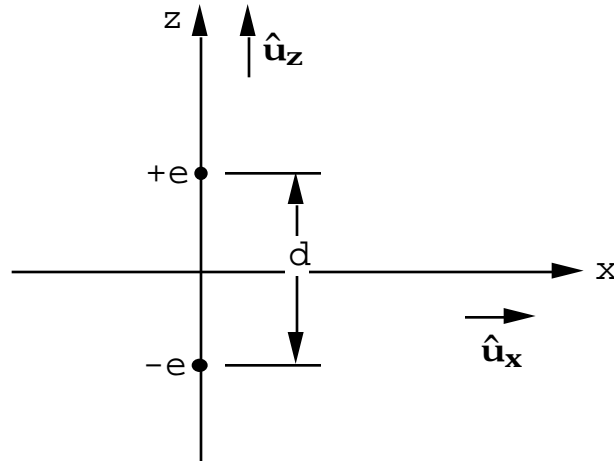
therefore

$$U = - \mathbf{m}_2 \cdot \mathbf{B}_1 = - \frac{\mu_0 m_1 m_2}{4 R^3} (2\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2).$$

This expression clearly has a minimum when $\cos\theta_1 = \cos\theta_2 = 1$ and $\sin\theta_1 = \sin\theta_2 = 0$, i.e. when $\theta_1 = \theta_2 = 0$ or π .

Problem (1.6)

A proton and an electron are separated by $10^{-12} \text{ m} = d$ as shown in the sketch.



- (a) Calculate the strength of the electric field 1 micron ($= 10^{-6}$ m) distant from a proton.
- (b) Calculate the strength of the electric field a = 1 micron from the above pt dipole at $\mathbf{r} = a\hat{\mathbf{u}}_z$. What is the direction of this electric field?
- (c) Calculate the strength of the electric field a distance $a = 1$ micron from the dipole at the point $\mathbf{r} = -a\hat{\mathbf{u}}_z$. What is the direction of this electric field?
- (d) Calculate the strength and direction of the electric field at $\mathbf{r} = a\hat{\mathbf{u}}_x$ where $a = 1$ micron.
- (e) Calculate the strength and direction of the electric field for the above dipole at $\mathbf{r} = \frac{1}{2} a(\hat{\mathbf{u}}_x + \hat{\mathbf{u}}_z)$ and $a = 1$ micron.
- N.B.** $\hat{\mathbf{u}}_x, \hat{\mathbf{u}}_y, \hat{\mathbf{u}}_z$ are unit vectors along x, y, z .

Answer (1.6)

a) $|\mathbf{E}| = \frac{1}{4\pi\epsilon_0} \frac{e}{r^2}$

$$\frac{1}{4 \pi \epsilon_0} = 9 \times 10^9 \quad e = 1.6 \times 10^{-19} \text{ Coulombs}$$

$$r = 10^{-6} \text{ m.}$$

$$|\mathbf{E}| = \frac{(9 \times 10^9) (1.6 \times 10^{-19})}{10^{-12}} = \underline{1440} \text{ Volts/m.}$$

b) For a point dipole $\mathbf{E}_d = \frac{1}{4 \pi \epsilon_0} \left[\frac{3(\mathbf{p} \cdot \mathbf{r})\mathbf{r}}{r^5} - \frac{\mathbf{p}}{r^3} \right]$

In this case \mathbf{p} and \mathbf{r} are both along z and hence parallel

$$\begin{aligned} |\mathbf{E}_d| &= \frac{1}{4 \pi \epsilon_0} \left(\frac{2p}{r^3} \right) \\ &= \frac{1}{4 \pi \epsilon_0} \left(\frac{e}{r^2} \right) \left(\frac{2d}{r} \right) \\ &= \frac{(2)(10^{-12})}{10^{-6}} = (2 \times 10^{-6}) \times \text{part (a)} \end{aligned}$$

So $|\mathbf{E}_d| = 2.88 \times 10^{-3} \text{ Volts/m}$ and directed along z .

c) For this part $\mathbf{p} \cdot \mathbf{r} = -e d a$

$$\text{since } \mathbf{p} = e d \hat{\mathbf{u}}_z$$

$$\text{and } \mathbf{r} = -a \hat{\mathbf{u}}_z$$

$$\text{therefore } (\mathbf{p} \cdot \mathbf{r})\mathbf{r} = e a^2 d \hat{\mathbf{u}}_z$$

$$\text{So } \frac{(\mathbf{p} \cdot \mathbf{r})\mathbf{r}}{r^5} = \frac{e d}{a^3} \hat{\mathbf{u}}_z$$

$$\text{and } |\mathbf{E}_d| = \frac{1}{4 \pi \epsilon_0} \left| \left(\frac{3e d}{a^3} \hat{\mathbf{u}}_z - \frac{e d}{a^3} \hat{\mathbf{u}}_z \right) \right| = \frac{1}{4 \pi \epsilon_0} \frac{2e d}{a^3}$$

or exactly the same as part (b). The electric field is also directed along z, just as in part (b).

(d) Here $\mathbf{p} \cdot \mathbf{r} = 0$ because \mathbf{p} is directed along z whereas \mathbf{r} is directed along x.

$$\begin{aligned} \text{Therefore } \mathbf{E}_d &= \frac{-1}{4 \pi \epsilon_0} \frac{\mathbf{p}}{r^3} \\ &= -\left(\frac{e}{4 \pi \epsilon_0} \right) \frac{1}{a^2} (d/a) \hat{\mathbf{u}}_z \end{aligned}$$

i.e. directed along -z and half as large as the electric field for a point along the dipole axis and a meters from the dipole.

$$|\mathbf{E}_d| = 1.44 \times 10^{-3} \text{ Volts/m}$$

$$\begin{aligned} \text{e) } \mathbf{p} &= ed\hat{\mathbf{u}}_z & \mathbf{p} \cdot \mathbf{r} &= \frac{eda}{2} \\ \mathbf{r} &= \frac{a}{2} (\hat{\mathbf{u}}_x + \hat{\mathbf{u}}_z) \end{aligned}$$

$$\mathbf{E}_d = \frac{1}{4 \pi \epsilon_0} \left[\frac{3ed}{(2)(2)} \frac{\hat{\mathbf{u}}_x + \hat{\mathbf{u}}_z}{a^3} - \frac{ed\hat{\mathbf{u}}_z}{a^3} \right]$$

$$\begin{aligned} \mathbf{E}_d &= \frac{e}{4 \pi \epsilon_0} \frac{d}{a^3} \left[\frac{3}{2} (\hat{\mathbf{u}}_x + \hat{\mathbf{u}}_z) - \hat{\mathbf{u}}_z \right] \\ &= \frac{e}{4 \pi \epsilon_0} \left(\frac{1}{a^2} \right) \left(\frac{d}{2a} \right) (3\hat{\mathbf{u}}_x + \hat{\mathbf{u}}_z) \end{aligned}$$

So \mathbf{E}_d is directed 18.4° from the xy plane and has the magnitude

$$|\mathbf{E}_d| = (1.58 \times 10^{-6}) (1440) = \underline{2.28 \times 10^{-3}} \text{ Volts/m.}$$

Problem (1.7).

Show that the magnetic field at the center of a uniformly magnetized sphere containing a small hole at the center is zero. Uniform magnetization means \mathbf{M} is constant. Without loss of generality, one can take the magnetization to be directed along the z-axis, ie $\mathbf{M} = M_0 \mathbf{u}_z$.

(Hint: Add up all the contributions to the field at the center due to volume elements at a distance r from the center. In polar co-ordinates $d = r^2 dr \sin \theta d\theta$, and $d\mathbf{m} = M_0 d \mathbf{u}_z$.)

Answer (1.7).

If $\mathbf{r} = -x\mathbf{u}_x - y\mathbf{u}_y - z\mathbf{u}_z$ then $\mathbf{m} \cdot \mathbf{r} = -M_0 d z$ (remember that \mathbf{r} is the vector drawn from the magnetic moment to the point of observation).

$$\mathbf{B} = \frac{\mu_0}{4} \frac{3(\mathbf{m} \cdot \mathbf{r})\mathbf{r}}{r^5} - \frac{\mathbf{m}}{r^3}, \text{ so that}$$

$$B_x = \frac{\mu_0}{4} \frac{M_0 d}{r^5} (3xz), \quad B_y = \frac{\mu_0}{4} \frac{M_0 d}{r^5} (3yz),$$

$$B_z = \frac{\mu_0}{4} \frac{M_0 d}{r^5} (2z^2 - x^2 - y^2),$$

Convert to polar co-ordinates and integrate over θ from 0 to π , and over ϕ from 0 to 2π . All field components integrate to zero.

Problem (1.8)

The fields generated at the position \mathbf{r} from a slowly moving, spinless, point charge are given by

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0} \frac{\mathbf{r}}{r^3} \quad \text{and} \quad c\mathbf{B} = \frac{\mathbf{v}}{c} \times \mathbf{E}.$$

Consider a particle moving in a circular orbit whose position at time t is given by

$$\mathbf{a} = a \cos t \hat{\mathbf{u}}_x + \sin t \hat{\mathbf{u}}_y.$$

(a) Show that the time averaged electric field seen by an observer at $\mathbf{R} = X \hat{\mathbf{u}}_x + Y \hat{\mathbf{u}}_y + Z \hat{\mathbf{u}}_z$ is given by

$$\langle \mathbf{E}_p \rangle = \frac{q}{4\pi\epsilon_0} \left(\frac{\mathbf{R}}{R^3} \right) \text{ to terms of order } (a/R)^2.$$

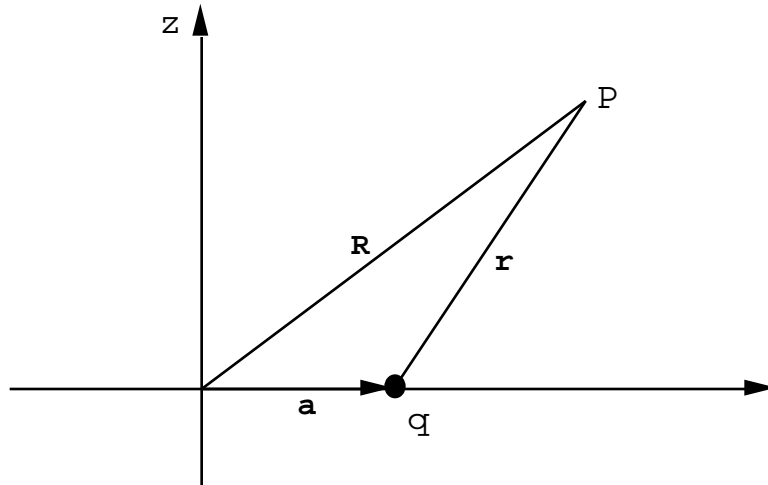
(b) Show that to lowest order in (a/R) the magnetic field observed at \mathbf{R} is given by

$$\langle \mathbf{B}_p \rangle = \frac{1}{c^2} \frac{1}{4\pi\epsilon_0} \left[\frac{3z\mathbf{R}}{R^5} - \frac{\hat{\mathbf{u}}_z}{R^3} \right] \left(\frac{qa^2}{2} \right)$$

or since $\mathbf{m} = \left(\frac{qa^2}{2} \right) \hat{\mathbf{u}}_z$ is the magnetic moment ($|\mathbf{m}| = I a^2$ where I is the current in Amps)

$$\langle \mathbf{B}_p \rangle = \frac{\mu_0}{4\pi} \left[3 \frac{\mathbf{m} \cdot \mathbf{R}}{R^5} \mathbf{R} - \frac{\mathbf{m}}{R^3} \right]$$

$$\text{and } c^2 = \frac{1}{\epsilon_0 \mu_0}.$$

Answer (1.8)

We have $\mathbf{r} + \mathbf{a} = \mathbf{R}$

$$\mathbf{r} = \mathbf{R} - \mathbf{a}$$

where $\mathbf{R} = X \hat{\mathbf{u}}_x + Y \hat{\mathbf{u}}_y + Z \hat{\mathbf{u}}_z$

and $\mathbf{a} = a \cos t \hat{\mathbf{u}}_x + a \sin t \hat{\mathbf{u}}_y$

$$r^2 = (X - a \cos t)^2 + (Y - a \sin t)^2 + Z^2$$

$$r^2 = X^2 + Y^2 + Z^2 + a^2 - 2aX \cos t - 2aY \sin t$$

$$\text{or } r^2 = R^2 \left[1 + \left(\frac{a}{R}\right)^2 - \frac{2aX}{R^2} \cos t - \frac{2aY}{R^2} \sin t \right].$$

Keep only the lowest terms in $\left(\frac{a}{R}\right)$:

$$r \cong R \left[1 - \frac{2aX}{R^2} \cos t - \frac{2aY}{R^2} \sin t \right]^{1/2}$$

$$\text{So } \frac{1}{r^3} \cong \frac{1}{R^3} \left[1 - \frac{2aX}{R^2} \cos t - \frac{2aY}{R^2} \sin t \right]^{-3/2}$$

$$\frac{1}{r^3} \cong \frac{1}{R^3} \left[1 + \frac{3aX}{R^2} \cos t + \frac{3aY}{R^2} \sin t \right]$$

$$\mathbf{E}_P = \frac{q}{4 \pi \epsilon_0} \left(\frac{\mathbf{r}}{r^3}\right) = \frac{q}{4 \pi \epsilon_0} \frac{(\mathbf{R} - \mathbf{a})}{R^3} \left[1 + \frac{3aX}{R^2} \cos t + \frac{3aY}{R^2} \sin t \right]$$

Now multiply out and take time averages.

$$\langle \cos t \rangle = \langle \sin t \rangle = \langle \sin t \cos t \rangle = 0$$

$$\langle \cos^2 t \rangle = \langle \sin^2 t \rangle = 1/2.$$

Notice that all terms proportional to a average to zero.

$$(a) \quad \langle \mathbf{E}_p \rangle \cong \frac{q}{4\pi\epsilon_0} \left(\frac{\mathbf{R}}{R^3} \right). \text{ The correction terms are of order } \left(\frac{a}{R} \right)^2.$$

$$(b) \quad \mathbf{B}_p = \frac{1}{c^2} (\mathbf{v} \times \mathbf{E})$$

$$\mathbf{v} = \frac{d\mathbf{a}}{dt} = -a \sin t \hat{\mathbf{u}}_x + a \cos t \hat{\mathbf{u}}_y$$

$$\mathbf{B}_p \cong \frac{1}{c^2} \left(\frac{q}{4\pi\epsilon_0} \right) \frac{[(\mathbf{v} \times \mathbf{R}) - (\mathbf{v} \times \mathbf{a})]}{R^3} \left\{ 1 + \frac{3aX}{R^2} \cos t + \frac{3aY}{R^2} \sin t \right\}$$

$$\begin{aligned} \mathbf{v} \times \mathbf{R} &= (aZ \cos t) \hat{\mathbf{u}}_x + (aZ \sin t) \hat{\mathbf{u}}_y \\ &\quad - [aY \sin t + aX \cos t] \hat{\mathbf{u}}_z \end{aligned}$$

$$\mathbf{v} \times \mathbf{a} = -a^2 \hat{\mathbf{u}}_z$$

Multiply out the terms in \mathbf{B}_p and take time averages. The result is

$$\begin{aligned} \langle \mathbf{B}_p \rangle &= \frac{1}{c^2} \left(\frac{q}{4\pi\epsilon_0} \right) \frac{1}{R^3} \left\{ \left(\frac{3a^2}{2R^2} \right) \overline{XZ} \hat{\mathbf{u}}_x + \left(\frac{3a^2}{2R^2} \right) \overline{YZ} \hat{\mathbf{u}}_y - \right. \\ &\quad \left. \left(\frac{3a^2}{2R^2} \right) \overline{(X^2 + Y^2)} \hat{\mathbf{u}}_z + a^2 \hat{\mathbf{u}}_z \right\} \end{aligned}$$

add and subtract $\frac{3a^2}{2R^2} Z^2 \hat{\mathbf{u}}_z$ to obtain

$$\langle \mathbf{B}_p \rangle = \frac{1}{c^2} \left(\frac{q}{4\pi\epsilon_0} \right) \frac{1}{R^3} \left\{ \left(\frac{3a^2}{2R^2} \right) Z \mathbf{R} - \left(\frac{a^2}{2} \right) \hat{\mathbf{u}}_z \right\}.$$

$$\text{Now } \frac{1}{c^2} = \mu_0 \text{ and } \mathbf{m} = \frac{qa^2}{2} \hat{\mathbf{u}}_z$$

$$\text{Therefore } \langle \mathbf{B}_p \rangle = \left(\frac{\mu_0}{4} \right) \left[\frac{3(\mathbf{m} \cdot \mathbf{R})\mathbf{R}}{R^5} - \frac{\mathbf{m}}{R^3} \right].$$

Problem (1.9)

Given the following scalar functions, V , expressed in cylindrical polar co-ordinates. For each function calculate

(1) the components of $\text{grad } V$

(2) $\nabla^2 V$

(a) $V = r \cos \theta$

(b) $V = \ln r$

(c) $V = \frac{\cos \theta}{r}$

(d) $V = \frac{\cos n\theta}{r^n}$, where n is an integer either positive or negative.

Answer (1.9)

$$\text{grad}V = \left(\frac{\partial V}{\partial r}\right) \hat{u}_r + \frac{1}{r} \left(\frac{\partial V}{\partial \theta}\right) \hat{u}_\theta + \left(\frac{\partial V}{\partial z}\right) \hat{u}_z.$$

$$\nabla^2 V = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2}$$

(a) $V = r \cos \theta$. $\text{grad}V|_r = \cos \theta$
 $\text{grad}V|_\theta = -\sin \theta$

These correspond to a unit vector along x !

$$\nabla^2 V = 0$$

(b) $V = \ln r$. $\text{grad}V|_r = \frac{1}{r}$
 $\text{grad}V|_\theta = 0$

$$\nabla^2 V = 0$$

(c) $V = \frac{\cos \theta}{r}$ $\text{grad}V|_r = -\frac{\cos \theta}{r^2}$

$$\text{grad}V|_\theta = -\frac{\sin \theta}{r^2}$$

$$\nabla^2 V = 0$$

$$(d) \quad V = \frac{\text{Cos}n}{r^n}. \quad \text{grad}V|_r = - \frac{n\text{Cos}n}{r^{n+1}}$$

$$\text{grad}V|_{\theta} = - \frac{n\text{Sin}(n)}{r^{n+1}}$$

$$\Delta V = 0 \quad \text{for any } n.$$

Problem (1.10)

Given the following scalar functions V expressed in spherical polar co-ordinates. For each function calculate

- (1) the components of $\text{grad } V$
- (2) ΔV

(a) $V = r \text{Cos}$

(b) $V = \frac{\text{Cos}}{r^2}$

(c) $V = r^2 (3\text{Cos}^2 - 1)$

(d) $V = \frac{(3\text{Cos}^2 - 1)}{r^3}$

(e) $V = \frac{\text{Cos}n}{r^n}$ where n is a positive integer.

Answer (1.10)

$$\nabla V = \left(\frac{\partial V}{\partial r}\right) \hat{u}_r + \frac{1}{r} \left(\frac{\partial V}{\partial \theta}\right) \hat{u}_\theta + \frac{1}{r \sin \theta} \left(\frac{\partial V}{\partial \phi}\right) \hat{u}_\phi$$

$$\Delta V = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2}$$

(a) $V = r \text{Cos} \quad \text{grad}V|_r = \text{Cos}$

$\text{grad}V|_{\theta} = - \text{Sin}$

$$\text{grad}V|_{\theta} = 0$$

These correspond to a constant field \hat{u}_z .

$$\Delta V = 0.$$

$$(b) \quad V = \frac{\cos \theta}{r^2}$$

$$\text{grad}V|_r = -\frac{2\cos \theta}{r^3} \quad \text{grad}V|_{\theta} = -\frac{\sin \theta}{r^3}$$

$$\text{grad}V|_{\phi} = 0$$

Corresponds to a dipole field.

$$\Delta V = 0.$$

$$(c) \quad V = r^2 (3\cos^2 \theta - 1) \quad \text{grad}V|_r = 2r (3\cos^2 \theta - 1)$$

$$\text{grad}V|_{\theta} = -6r \sin \theta \cos \theta$$

$$\text{grad}V|_{\phi} = 0$$

$$\Delta V = 0$$

$$(d) \quad V = \frac{3\cos^2 \theta - 1}{r^3} \quad \text{grad}V|_r = -\frac{3}{r^4} (3\cos^2 \theta - 1)$$

$$\text{grad}V|_{\theta} = -\frac{6}{r^4} \sin \theta \cos \theta$$

$$\text{grad}V|_{\phi} = 0$$

$$\Delta V = 0$$

$$(e) \quad V = \frac{\cos n\theta}{r^n} \quad \text{grad}V|_r = -\frac{n\cos(n\theta)}{r^{n+1}}$$

$$\text{grad}V|_{\theta} = -\frac{n\sin(n\theta)}{r^{n+1}} \quad \text{grad}V|_{\phi} = 0$$

$$\Delta V = -\frac{n}{r^{n+2}} (\sin \theta \cos(n\theta) + \cos \theta \sin(n\theta)).$$

Does not equal zero!

Problem (1.11)

Calculate the vector field $\mathbf{B} = \text{curl}\mathbf{A}$ for the following fields, \mathbf{A} .

(a) In cylindrical polar co-ordinates

$$A_r = 0$$

$$A_\phi = 0$$

$$A_z = -\frac{\mu_0 I}{2} \ln r$$

(b) In cylindrical polar co-ordinates

$$A_r = 0$$

$$A_\phi = \frac{B_0 r}{2}$$

$$A_z = 0$$

(c) $\mathbf{A} = \frac{\mu_0}{4} \left(\frac{\mathbf{m} \times \mathbf{r}}{r^3} \right)$, where $\mathbf{m} = m_0 \hat{\mathbf{u}}_z$.

Show that in spherical polar co-ordinates if $\mathbf{m} = m_0 \hat{\mathbf{u}}_z$ then

$A_r = A_\phi = 0$ and $A_z = \frac{\mu_0}{4} \frac{m_0 \sin \theta}{r^2}$. This can be used to calculate

$\text{curl } \mathbf{A}$.

Answer (1.11)

$$(a) \quad \mathbf{B} = \text{curl}\mathbf{A} = \frac{1}{r} \begin{vmatrix} \hat{\mathbf{u}}_r & r\hat{\mathbf{u}}_\phi & \hat{\mathbf{u}}_z \\ \frac{1}{r} & - & \frac{1}{z} \\ 0 & 0 & A_z \end{vmatrix} = \begin{vmatrix} \frac{1}{r} & A_z \\ - & \frac{A_z}{r} \\ 0 & 0 \end{vmatrix}$$

$$\text{But } \frac{A_z}{r} = 0 \quad B_r = 0$$

$$B = \frac{\mu_0 I}{2r}$$

$$B_z = 0$$

The field due to a current I Amps flowing along a long wire oriented along z .

$$(b) \quad \mathbf{B} = \text{curl} \mathbf{A} = \frac{1}{r} \begin{vmatrix} \hat{\mathbf{u}}_r & r\hat{\mathbf{u}} & \hat{\mathbf{u}}_z \\ \frac{1}{r} & 0 & 0 \\ 0 & rA & 0 \end{vmatrix} = \begin{vmatrix} -\frac{A}{z} \\ 0 \\ \frac{1}{r} \frac{(rA)}{r} \end{vmatrix}$$

$$\text{But } \frac{A}{z} = 0 \quad \text{and } A = \frac{B_0 r}{2}$$

$$B_r = 0$$

$$B_\theta = 0$$

$$B_z = B_0$$

This is the field inside an infinitely long solenoid.

$$(c) \quad \mathbf{A} = \frac{\mu_0}{4} \frac{(\mathbf{m} \times \mathbf{r})}{r^3}$$

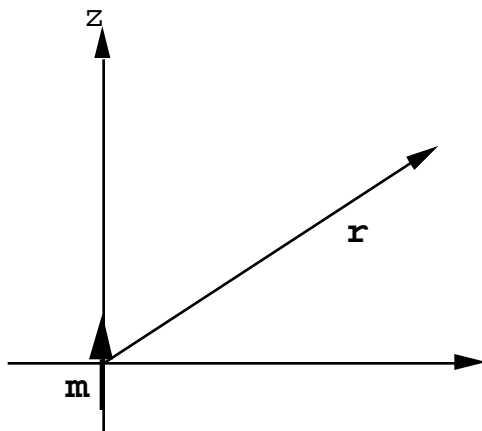
If $\mathbf{m} = m_0 \hat{\mathbf{u}}_z$ this generates the field due to a magnetic dipole.

For $\mathbf{m} = m_0 \hat{\mathbf{u}}_z$ $(\mathbf{m} \times \mathbf{r})$ is a vector in the direction having the magnitude $m_0 r \sin \theta$.

$$A_r = 0$$

$$A_\theta = 0$$

$$A_\phi = \frac{\mu_0}{4} \frac{m_0}{r^2} \sin \theta, \quad (\text{see the figure}).$$

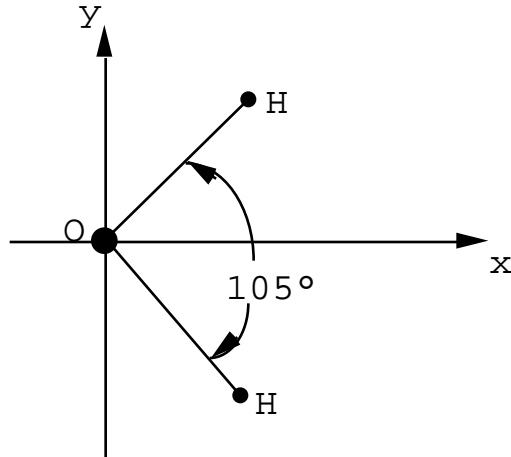


$$\mathbf{B} = \text{curl} \mathbf{A} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \hat{\mathbf{u}}_r & r \hat{\mathbf{u}}_\theta & r \sin \theta \hat{\mathbf{u}}_\phi \\ \frac{1}{r} & -\frac{\cos \theta}{r} & 0 \\ 0 & 0 & r \sin \theta \end{vmatrix} = \begin{vmatrix} \frac{1}{r^2 \sin \theta} & \frac{(r \sin \theta) \cos \theta}{r^2 \sin \theta} \\ -\frac{1}{r \sin \theta} & \frac{(r \sin \theta)}{r} \\ 0 & 0 & 0 \end{vmatrix}$$

$$B_r = \frac{\mu_0}{4} \frac{2m_0 \cos \theta}{r^3}, \quad B_\theta = \frac{\mu_0}{4} \frac{m_0 \sin \theta}{r^3}, \quad B_\phi = 0.$$

Problem (1.12)

A water molecule is planar but the angle between the two oxygen-hydrogen bonds is 105° as shown in the sketch.



(a) If the charge on the oxygen is twice the electronic charge i.e. $-2|e|$ and the charge on each hydrogen is $q_H = +|e|$, calculate the dipole moment of the molecule assuming an O-H bond length of 5×10^{-10} m.
 [The measured dipole moment is $p = 6.17 \times 10^{-30}$ Coulomb-m].

(b) If all of the dipoles in a cubic meter of water were aligned what would be the resulting density of electric dipoles $|P|$?

$$\text{Use } p = 6.17 \times 10^{-30} \text{ cm.}$$

Answer (1.12)

(a) The dipole moment is $p = qd$. In the H_2O molecule

$$q = 2|e| = (2)(1.60 \times 10^{-19}) \text{ Coulombs}$$

$$\text{or } q = 3.2 \times 10^{-19} \text{ C}$$

The distance $d = b \cos\left(\frac{105}{2}\right)$ where $b = 5 \times 10^{-10}$ m is

the bond length; $d = 3.04 \times 10^{-10}$ m

$$p = (3.2)(3.04) \times 10^{-29} \text{ Coulomb m} = \underline{9.74 \times 10^{-29} \text{ Cm}}$$

Compared with experiment this is too large by ~ 15.7 times.

(b) The molar volume of H_2O is 18 c.c.

$$\text{No. of moles in } 1 \text{ m}^3 = 10^6/18 = 5.56 \times 10^4 \text{ moles.}$$

$$\begin{aligned} \text{No. of molecules in } 1 \text{ m}^3 &= (6.02 \times 10^{23})(5.56 \times 10^4) = \\ &3.34 \times 10^{28} \text{ molecules.} \end{aligned}$$

$$|\mathbf{P}| = (3.34 \times 10^{28})(6.17 \times 10^{-30}) = \underline{0.21 \text{ Coulombs/m}^2}.$$

(This is very large--in fact H_2O has no permanent dipole moment because the molecules are oriented at random).

Problem (1.13)

An iron atom in metallic iron carries a magnetic moment of 2.2 Bohr magnetons. (1 Bohr magneton, μ_B , is

$\mu_B = 9.27 \times 10^{-24} \text{ Amp m}^2$ (= Joules/Tesla)). The density of iron is 7.87 gms/cc and its molecular weight is 55.85 gms.

If all of the atomic moments were aligned parallel what would be the magnetization per unit volume of iron? Compare this value with the observed internal magnetic field of saturated iron at room temperature $|\mathbf{B}| = \mu_0 |\mathbf{M}| = 2.15 \text{ Teslas}$

$$= 2.15 \text{ Webers/m}^2.$$

Answer (1.13)

The molar volume of iron is $\frac{55.85}{7.87} = 7.10 \text{ cc}$.

The number of atoms in $/\text{m}^3$ is

$$N = (6.02 \times 10^{23}) \left(\frac{10^6}{7.10} \right) = 0.848 \times 10^{29} \text{ atoms/m}^3.$$

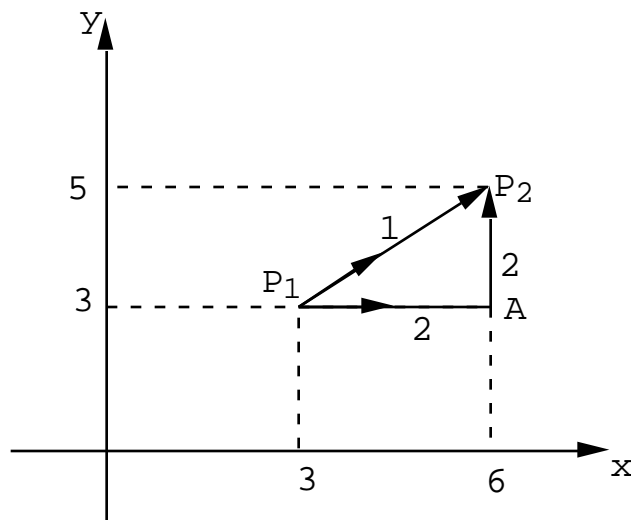
The magnetization/ m^3 $|\mathbf{M}| = (N)(2.2) \mu_B$

$$|\mathbf{M}| = \underline{0.173 \times 10^7 \text{ Amps/m}}.$$

This would give an internal field $|\mathbf{B}| = \mu_0 |\mathbf{M}|$ of $|\mathbf{B}| = (4 \times 10^{-7})(0.173 \times 10^7) = 2.17$ Teslas.

This means that at room temperature the fraction of aligned spins in iron is $\frac{2.15}{2.17} = 0.989$ i.e. Very nearly completely aligned!

Problem (1.14)



Given a vector function $\mathbf{F} = xy\hat{\mathbf{u}}_x + (3x-y^2)\hat{\mathbf{u}}_y$ evaluate the line integral from P_1 to P_2 along

- the direct path (1).
- the indirect path $P_1 \rightarrow A \rightarrow P_2$ (path (2)).

Answer (1.14)

The line P_1P_2 can be written

$\mathbf{s} = (3\hat{\mathbf{u}}_x + 3\hat{\mathbf{u}}_y) + (3\hat{\mathbf{u}}_x + 2\hat{\mathbf{u}}_y)L$ where L varies from $L = 0$ to $L = 1$. $L = 0$ corresponds to $P_1 (3\hat{\mathbf{u}}_x + 3\hat{\mathbf{u}}_y)$ whereas $L = 1$ corresponds to $P_2 (6\hat{\mathbf{u}}_x + 5\hat{\mathbf{u}}_y)$.

So $d\mathbf{s} = (3\hat{\mathbf{u}}_x + 2\hat{\mathbf{u}}_y)dL$ or $dx = 3dL$ and $dy = 2dL$.

(a) Now $\mathbf{F} \cdot d\mathbf{s} = 3xy dL + 2(3x - y^2) dL$

$$\int_{P_2}^{P_1} \mathbf{F} \cdot d\mathbf{s} = \int_{P_2(L=0)}^{P_1(L=1)} [3xy + 6x - 2y^2] dL$$

But $x = (3 + 3L)$ $y = 3 + 2L$ along the line
(components of \mathbf{s})

$$\begin{aligned} \int_{P_2}^{P_1} \mathbf{F} \cdot d\mathbf{s} &= \int_0^1 dL \left\{ 3(3 + 3L)(3 + 2L) + 6(3 + 3L) - \right. \\ &\quad \left. 2(3 + 2L)^2 \right\} \\ \int_{P_2}^{P_1} \mathbf{F} \cdot d\mathbf{s} &= \int_0^1 dL [27 + 39L + 10L^2] \\ &= 27 + (137/6) = \frac{299}{6}. \end{aligned}$$

(b) Along path (2)

$$\begin{aligned} \int_2^6 \mathbf{F} \cdot d\mathbf{s} &= \int_3^6 F_x(y=3) dx + \int_3^5 F_y(x=6) dy \\ &= 3 \int_3^6 x dx + \int_3^5 (18 - y^2) dy \\ &= \left(\frac{3}{2} \right) (27) + 36 - \frac{98}{3} = 36 + \left(\frac{47}{6} \right) = \frac{263}{6} \end{aligned}$$

The line integral is different for the two paths.

Therefore \mathbf{F} is not a conservative field. Indeed,

$$\text{curl } \mathbf{F} = \begin{vmatrix} 0 \\ 0 \\ 3x \end{vmatrix} \text{ and therefore curl } \mathbf{F} \text{ does not vanish}$$

everywhere.

Problem (1.15)

Given the vector function $\mathbf{E} = y\hat{\mathbf{u}}_x + x\hat{\mathbf{u}}_y$. Evaluate the line integral $\int_1^2 \mathbf{E} \cdot d\mathbf{L}$ from $P_1 (2,1,-1)$ to $P_2 (8,2,-1)$

- along the parabola $x = 2y^2$,
- along the straight line joining the two points.
- Is \mathbf{E} a conservative vector field?

Answer (1.15)

$$\text{curl}\mathbf{E} = \begin{vmatrix} \hat{\mathbf{u}}_x & \hat{\mathbf{u}}_y & \hat{\mathbf{u}}_z \\ x & y & z \\ y & x & 0 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ (1-1)=0 \end{vmatrix} = 0.$$

Therefore \mathbf{E} is a conservative vector field.

$$\begin{aligned} \text{(a)} \quad \int_1^2 \mathbf{E} \cdot d\mathbf{L} &= \int_1^2 E_x dx + \int_1^2 E_y dy \\ &= \int_2^8 y dx + \int_1^2 x dy \end{aligned}$$

But $y = x/2$ $x = 2y^2$ along the parabola

$$\begin{aligned} \int_1^2 \mathbf{E} \cdot d\mathbf{L} &= \int_2^8 \frac{x^{1/2} dx}{2} + 2 \int_1^2 y^2 dy = \left. \frac{2x^{3/2}}{3} \right|_2^8 + \left. \frac{2y^3}{3} \right|_1^2 \\ &= \frac{1}{3} [2^5 - 2^2 + 14] = 42/3 = 14. \end{aligned}$$

(b) Since $\text{curl } \mathbf{E} = 0$ the line integral along the second path must also be equal to 14.

Check Let $\mathbf{r}_1 = 2\hat{\mathbf{u}}_x + \hat{\mathbf{u}}_y - \hat{\mathbf{u}}_z$ (the vector to P_1)

Let $\mathbf{r}_2 = 8\hat{\mathbf{u}}_x + 2\hat{\mathbf{u}}_y - \hat{\mathbf{u}}_z$ (the vector to P_2).

Then any point on the straight line from P_1 to P_2 can be specified by

$$\mathbf{L} = \mathbf{r}_1 + L (\mathbf{r}_2 - \mathbf{r}_1)$$

where L runs from $L = 0$ (P_1) to $L = 1$ (P_2)

$$\mathbf{L} = (2\hat{\mathbf{u}}_x + \hat{\mathbf{u}}_y - \hat{\mathbf{u}}_z) + (6\hat{\mathbf{u}}_x + \hat{\mathbf{u}}_y)$$

$$d\mathbf{L} = (6\hat{\mathbf{u}}_x + \hat{\mathbf{u}}_y)dL$$

$$\mathbf{E} \cdot d\mathbf{L} = 6y dL + x dL$$

However, along the st. line \mathbf{L} $x = 2 + 6L$

$$y = 1 + L$$

$$\mathbf{E} \cdot d\mathbf{L} = 6(1 + L) dL + (2 + 6L) dL = (8 + 12L) dL$$

$$\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{L} = \int_0^1 (8 + 12L) dL = 8 + 6 = 14 \quad \underline{\text{Q.E.D.}}$$