CHAPTER 12 PROBLEMS and SOLUTIONS

Problem(12.1). Microwave power of 1 Watt at a frequency of 24 GHz is transmitted through a piece of rectangular waveguide whose inside dimensions are 1 cm x 0.5 cm. Let the z-axis lie parallel with the waveguide axis, and let the microwaves be propagating in the +z direction. Use $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$.

(a) Write expressions for the electric and magnetic fields in the waveguide if the time variation is $e^{-i\omega t}$.

(b) Calculate the amplitudes of the electric and magnetic field components.

(c) Calculate the time-averaged energy density contained in the fields.

(d) With what velocity is the above energy density transported along the waveguide?

(e) Show that the magnetic field vector rotates with time at points which are part way across the width of the waveguide. Show that for points near x=a/4 the rotation is clockwise when viewed from a point on the plus y-axis and looking towards the x-z plane, whereas the rortation is counter-clockwise near x=3a/4.

Answer(12.1).

(a) For a frequency F= 24 GHz, $\omega = 2\pi F = 1.508\times10^{11}$ radians/sec. For the TE_{10} mode (all other modes are cut-off)

$$\mathbf{E_y = E_0 \; Sin(\frac{\pi x}{a}) \; e^{i(k_g z - \omega t)}} \;,$$

where the waveguide walls are at x=0,a and at y=0,b: there is no spatial variation along the narrow dimension of the guide. The field components must satisfy the wave equation: in particular,

$$\nabla^2 E_y = \varepsilon_0 \mu_0 \frac{\partial^2 E_y}{\partial t^2} \;,$$

from which

$$\left(\frac{\pi}{a}\right)^2 + k_g^2 = \left(\frac{\omega}{c}\right)^2 .$$

For the present case, $\frac{\pi}{a} = 314.2 \text{ m}^{-1}$

$$\frac{\omega}{c} = 502.7 \text{ m}^{-1}$$

so that

$$k_g = 392.4 \text{ m}^{-1}.$$

From curlE= $i\omega\mu_0\mathbf{H}$, using the fact that **E** has only a y-component, one finds

$$\mathbf{H_x = -\left(\frac{k_g}{\omega\mu_0}\right) Sin(\frac{\pi x}{a}) \; E_0 \; e^{i(k_g z - \omega t)}} \;,$$

and $i\omega\mu_0 H_z = \frac{\partial E_y}{\partial x} = \left(\frac{\pi}{a}\right) E_0 \, \mathrm{Cos}\left(\frac{\pi x}{a}\right) e^{i(k_g z - \omega t)}$,

or

$$\mathbf{H_z = \frac{-i\pi}{\mu_0 a} E_0 \, Cos\left(\frac{\pi x}{a}\right) e^{i(k_g z - \omega t)}}.$$

Note that $E_y = - Z_g H_x$ where $Z_g = - \left(\frac{\omega}{c k_g}\right) Z_0$, and

$Z_0 = \mu_0 c = 377$ Ohms.

(b) $$S_z = - E_y H_x \text{ Watts/m}^2.$$

$$<S_z> = -\frac{1}{2} \, \mathrm{Real}\left(E_y H_x^*\right) = \frac{1}{2} \frac{|E_0|^2}{|Z_g|} \mathrm{Sin}^2\left(\frac{\pi x}{a}\right).$$

The average across the guide is given by

$$<<S_z>> = \frac{1}{4} \frac{|E_0|^2}{\left(\frac{\omega}{c k_g}\right) Z_0} ,$$

where E_0 is the electric field amplitude. Now $Z_g = Z_0 \left(\frac{\omega}{c k_g}\right) = 482.9$ Ohms, and $<<S_z>>ab = 1$ Watt,

therefore $<<S_z>> = 2\text{x}10^4$ Watts/m^2,

so that **E_0 = 6216 Volts/meter**, or 31.1 Volts across the narrow dimension of the waveguide. The x-component of the magnetic field amplitude is **$|H_x|$ = 12.87 Amps/m.** The amplitude of the longitudinal magnetic field component is $\mathbf{|H_0| = 10.31}$ **Amps/m.**

(c) The time-averaged energy density contained in the fields is given by

$$<W> = <\varepsilon_0 E_y^2/2> + <\mu_0 H_x^2/2> + <\mu_0 H_z^2/2>,$$

or

$$<W> = \frac{\varepsilon_0 E_0^2 \mathrm{Sin}^2(\pi x/a)}{4} +$$

$$\frac{1}{4\mu_0}\left(\frac{k_g^2}{\omega^2} E_0^2\ \mathrm{Sin}^2(\pi x/a) + \frac{\pi^2}{a^2\omega^2} E_0^2\ \mathrm{Cos}^2(\pi x/a)\right).$$

Averaged over the guide cross-section, this expression gives

$$<<W>> = \varepsilon_0 \frac{E_0^2}{4}\ \text{Joules/m}^3 = 85.4\text{x}10^{-6}\ \text{J/m}^3.$$

(d) The group velocity is the rate of energy transport down the guide;

$$<<S_z>> = V_g <<W>>.$$

It follows from this that

$$V_g = c\,\frac{k_g}{(\omega/c)} = 0.781\ c = \mathbf{2.34x10^8\ m/sec.}$$

The group velocity is also given by $V_g = \dfrac{\partial\omega}{\partial k_g}$.

(e) Near x=a/4 $H_x = \frac{-k_g}{\mu_0\omega}\frac{E_0}{\sqrt{2}}\; e^{-i\omega t}$

$$H_z = \frac{\pi}{a\mu_0\omega}\frac{E_0}{\sqrt{2}}\; e^{-(i\omega t - \pi/2)},$$

therefore if $H_x = \frac{-k_g}{\mu_0\omega}\frac{E_0}{\sqrt{2}}\cos\omega t$,

then

$$H_z = -\frac{\pi}{a\mu_0\omega}\frac{E_0}{\sqrt{2}}\sin\omega t.$$

These expressions describe an elliptically polarized wave (nearly circularly polarized because $\frac{k_g}{(\pi/a)} = 1.25$) rotating in the direction from z to -x, i.e. clockwise looking from +y towards the x-z plane.

Similarly, near x=3a/4 $H_x = -\frac{k_g}{\mu_0\omega}\frac{E_0}{\sqrt{2}}\cos\omega t$, and

$$H_z = \frac{\pi}{a\mu_0\omega}\frac{E_0}{\sqrt{2}}\sin\omega t,$$

corresponding to a counter-clockwise rotation looking from +y towards the xz plane.

Problem(12.2). An attempt is made to propagate a 10 GHz microwave signal along a rectangular air-filled waveguide whose internal dimemsions are 1 cm x 0.50 cm. Use ε_0 and μ_0 for the dielectric constant and the permeability.

(a) Write expressions for the electric and magnetic fields associated with the non-propagating TE_{10} mode.

(b) Over what distance is the amplitude of the microwave fields attenuated by 1/e?

(c) Calculate the z-component of the Poynting vector and show that it corresponds to a periodic flow of energy across the waveguide section whose time average is zero.

Answer(12.2).

(a) F= 10 GHz $\omega = 6.28\times10^{10}$ rad./sec. $\frac{\omega}{c} = 2.094\times10^{2}\ \mathrm{m}^{-1}$.

$$\frac{\pi}{a} = 3.141\times10^{2}\ \mathrm{m}^{-1}.$$

For the TE_{10} mode $k_g^2 + \left(\frac{\pi}{a}\right)^2 = \left(\frac{\omega}{c}\right)^2$,

from which $k_g^2 = -5.4831\times10^{4}$, and $k_g = \pm i\ 2.342\times10^{2}\ \mathrm{m}^{-1}$, a pure imaginary number. Let $k_g = i\alpha$.

$$\mathbf{E_y = E_0\ Sin\left(\frac{\pi x}{a}\right)\ e^{-\alpha z}\ e^{-i\omega t}}$$

$$\mathbf{H_x = -\ \frac{i\alpha}{\omega\mu_0}\ E_0\ Sin\left(\frac{\pi x}{a}\right)\ e^{-\alpha z}\ e^{-i\omega t}}$$

$$\mathbf{H_z = -\ i\ \left(\frac{\pi}{a\omega\mu_0}\right)\ E_0\ Cos\left(\frac{\pi x}{a}\right)\ e^{-\alpha z}\ e^{-i\omega t}}.$$

(b) The attenuation length is $\frac{1}{\alpha} = \frac{10^{-2}}{2.34} = 4.27\times10^{-3}$ meters, or

$$\mathbf{1/\alpha = 4.27\ mm.}$$

(c) $S_z = -\ E_y H_x$, where for this problem

$$E_y = E_0 \, \mathrm{Sin}\left(\frac{\pi x}{a}\right) e^{-\alpha z} \, \mathrm{Cos}\omega t,$$

and

$$H_x = -\left(\frac{\alpha}{\omega\mu_0}\right) E_0 \, \mathrm{Sin}\left(\frac{\pi x}{a}\right) e^{-\alpha z} \, \mathrm{Sin}\omega t.$$

Therefore, $S_z = -E_y H_x = \dfrac{\alpha}{\omega\mu_0} E_0^2 \, \mathrm{Sin}^2\left(\dfrac{\pi x}{a}\right) e^{-2\alpha z} \, \mathrm{Sin}\omega t \mathrm{Cos}\omega t$

or $\boldsymbol{S_z = 1.483\times10^{-3} \, E_0^2 \, \mathrm{Sin}^2\left(\dfrac{\pi x}{a}\right) e^{-2\alpha z} \, \mathrm{Sin}2\omega t}$

since $\mathrm{Sin}\omega t \mathrm{Cos}\omega t = \frac{1}{2} \mathrm{Sin}2\omega t$.

Problem(12.3).

(a) Design a rectangular air-filled cavity to operate at 24 GHz in the TE_{103} mode. The cavity is to be constructed from a length of rectangular waveguide whose internal dimensions are 1 x 0.50 cm. Use ε_0 and μ_0 for the dielectric constant and the permeability.

(b) Write expressions for the fields in the cavity at resonance.

Answer(12.3).

(a) At 24 GHz $\omega = 1.508\times10^{11}$ rad./sec $\frac{\omega}{c} = 502.7\ m^{-1}$.

For the TE_{10} mode the guide wave-number can be calculated from

$$k_g^2 = \left(\frac{\omega}{c}\right)^2 - \left(\frac{\pi}{a}\right)^2$$

where a= 0.01 m is the broad dimension of the guide:

$$k_g = 3.925\times10^2\ m^{-1}.$$

The guide wavelength is $\lambda_g = 2\pi/k_g = 1.60\times10^{-2}$ m. = 1.60 cm.
The length of the cavity should be $L = \frac{3\lambda_g}{2}$ for the TE_{103} mode;

$$\mathbf{L = 2.40\times10^{-2}\ m. = 2.40\ cm.}$$

(b) For the forward propagating wave and a TE_{10} mode

$$E_y = E_0 \, \mathrm{Sin}\left(\frac{\pi x}{a}\right) e^{ik_g z} \, e^{-i\omega t},$$

For the backward propagating wave

$$E_y = E_0 \, \mathrm{Sin}\left(\frac{\pi x}{a}\right) e^{-ik_g z} \, e^{-i\omega t}.$$

In the cavity one must set up a standing wave along z which has nodes at z=0 and at $z = L = \frac{3\lambda_g}{2}$; i.e.

$$\mathbf{E_y = E_0 \, Sin\left(\frac{\pi x}{a}\right) Sin\left(\frac{n\pi z}{L}\right) Cos\omega t.}$$

From this electric field one can calculate the other field components using curl$\mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}$. For the TE_{10} mode the electric field has only one component, E_y, and

$$\frac{\partial E_y}{\partial z} = \mu_0 \frac{\partial H_x}{\partial t} \qquad (1)$$

$$\frac{\partial E_y}{\partial x} = -\mu_0 \frac{\partial H_z}{\partial t} \qquad (2)$$

From (1)

$$\mathbf{H_x = \left(\frac{1}{\mu_0 \omega}\right)\left(\frac{n\pi}{L}\right) E_0 \, Sin\left(\frac{\pi x}{a}\right) Cos\left(\frac{n\pi z}{L}\right) Sin\omega t.}$$

From (2)

$$H_z = -\left(\frac{1}{\mu_0 \omega}\right)\left(\frac{\pi}{a}\right) E_0 \,\mathrm{Cos}\left(\frac{\pi x}{a}\right) \mathrm{Sin}\left(\frac{n\pi z}{L}\right) \mathrm{Sin}\,\omega t.$$

For resonance $k_g = 3\lambda/2$ and therefore L= 2.40 cm.

Problem(12.4). A rectangular waveguide is filled with material characterized by a relative dielectric constant $\varepsilon_r = 9.00$. The inside dimensions of the waveguide are a= 1 cm, b= 0.50 cm.

(a) Over what frequency interval would this guide support only the TE_{10} mode?

(b) Calculate the time-averaged energy density for the TE_{10} mode, and average the resulting expression over the guide cross-section. Let the amplitude of the electric field be $E_y = E_0$.

(c) Calculate the time-averaged value of the Poynting vector, and average the resulting expression over the guide cross-section. Let the amplitude of the electric field be $E_y = E_0$.

(d) A signal having an average power of 1 Watt is transmitted down the guide at a frequency of 7.5 GHz. Calculate (i) the wavelength along the guide, λ_g; (ii) the ratio of the guide wavelength to the free space wavelength for a 7.5 GHz plane wave; (iii) the group velocity, i.e. the velocity with which information can be transmitted down the guide; (iv) the amplitude of the electric field.

Answer(12.4).

(a) For the TE_{10} mode the fields have the form

$$E_y = E_0 \, \mathrm{Sin}\left(\frac{\pi x}{a}\right) e^{i(k_g z-\omega t)} ,$$

$$H_x = -\left(\frac{k_g}{\omega\mu_0}\right) E_0 \, \mathrm{Sin}\left(\frac{\pi x}{a}\right) e^{i(k_g z-\omega t)},$$

$$H_z = -\left(\frac{i}{\omega\mu_0}\right)\left(\frac{\pi}{a}\right) E_0 \, \mathrm{Cos}\left(\frac{\pi x}{a}\right) e^{i(k_g z-\omega t)},$$

where $$\omega^2\varepsilon\mu_0 = k_g^2 + \left(\frac{\pi}{a}\right)^2$$

or $$\varepsilon_r \left(\frac{\omega}{c}\right)^2 = k_g^2 + \left(\frac{\pi}{a}\right)^2 .$$

If a= 1 cm = 0.01 m $$\left(\frac{\pi}{a}\right)^2 = 9.870\mathrm{x}10^4 \ \mathrm{m}^{-2}.$$

The cut-off frequency corresponds to $k_g= 0$; i.e. $\sqrt{\varepsilon_r}\left(\frac{\omega}{c}\right) = \frac{\pi}{a}$. At cut-off $\frac{\omega}{c} = \frac{314.2}{\sqrt{\varepsilon_r}} = 104.7 \ \mathrm{m}^{-1}$,

or **F = 5.00 GHz**.

For the higher order modes, cut-off corresponds to the condition $k_g= 0$, so that

$$\varepsilon_r \left(\frac{\omega}{c}\right)^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 ,$$

where $\frac{\pi}{a} = 314.2\ \text{m}^{-1}$, and $\frac{\pi}{b} = 628.4\ \text{m}^{-1}$.

For	m=0	n=1	F_{01} = 10.00 GHz
	m=1	n=1	F_{11} = 11.18 Ghz
	m=1	n=2	F_{12} = 20.62 GHz
	m=2	n=0	F_{20} = 10.00 GHz.

This waveguide will support only the TE_{10} mode for frequencies in the interval 5.00 to 10.00 GHz.

(b) The time-averaged energy density is given by

$$\langle W\rangle = \langle \varepsilon E_y^2/2\rangle + \langle \mu_0 H_x^2/2\rangle + \langle \mu_0 H_z^2/2\rangle,$$

$$\langle W\rangle = \frac{\varepsilon_r\varepsilon_0}{4} E_0^2 \,\mathrm{Sin}^2\left(\frac{\pi x}{a}\right) + + \frac{1}{4\mu_0\omega^2} k_g^2 E_0^2 \,\mathrm{Sin}^2\left(\frac{\pi x}{a}\right) + \frac{1}{4\mu_0\omega^2}\left(\frac{\pi}{a}\right)^2 E_0^2 \,\mathrm{Cos}^2\left(\frac{\pi x}{a}\right).$$

Take the spatial average over the cross-section of the waveguide:

$$\langle\langle W\rangle\rangle = \left(\varepsilon_r + \frac{1}{\omega^2\varepsilon_0\mu_0}\left(k_g^2 + \left(\frac{\pi}{a}\right)^2\right)\right)\frac{\varepsilon_0 E_0^2}{8},$$

$$\boldsymbol{\langle\langle W\rangle\rangle = \frac{\varepsilon_r\varepsilon_0}{4} E_0^2 \ \textbf{Joules/m}^3}.$$

(c) $S_z = - E_y H_x$,

$$\boldsymbol{\langle S_z\rangle = \frac{k_g}{2\omega\mu_0} E_0^2 \,\textbf{Sin}^2\left(\frac{\pi x}{a}\right)}.$$

The average over the x co-ordinate gives

$$\mathbf{\langle\langle S_z\rangle\rangle = \frac{k_g}{4\omega\mu_0} E_0^2 \quad Watts/m^2.}$$

(d) The group velocity is such that $\langle\langle S_z\rangle\rangle = V_g \langle\langle W\rangle\rangle$, therefore

$$V_g = \frac{c}{\varepsilon_r}\left(\frac{k_g}{(\omega/c)}\right).$$

At 7.5 GHz $k_0 = \frac{\omega}{c} = 157.1\ m^{-1}$, and the free space wavelength is $\lambda_0 = 4.00$ cm. The waveguide wave-vector is given by

$$k_g^2 = 9k_0^2 - \left(\frac{\pi}{a}\right)^2 = 12.337\times10^4,$$

and

$$k_g = 3.513\times10^2\ m^{-1}.$$

From this, the guide wavelength is

(i) $\mathbf{\lambda_g = \frac{2\pi}{k_g} = 1.788\ cm}$, and

(ii) $\mathbf{\frac{\lambda_g}{\lambda_0} = 0.447}$.

(iii) $V_g = \frac{c}{9}\left(\frac{3.512}{1.571}\right) = \mathbf{0.745\times10^8\ meters/sec}$.

(iv) $\langle\langle S_z\rangle\rangle = \frac{k_g}{4\omega\mu_0} E_0^2 = \frac{1}{ab} = 2\times10^4$ Watts/m^2.

From this $$E_0^2 = 4\frac{(\omega/c)}{k_g}\,(377)(2\text{x}10^4) = 1.349\text{x}10^7,$$

so that $$\mathbf{E_0 = 3673\ Volts/m.}$$

Problem(12.5). It is desired to construct a cylindrical air-filled cavity which will resonate at 10 GHz in the TE_{01} doughnut mode (this is a very low loss mode which is often used to construct frequency meters). If the radius of the cavity is chosen to be R= 2.50 cm how long should the cavity be made?

Answer(12.5). For the TE_{01} mode the tangential component of the electric field, E_θ, is proportional to the Bessel function $J_0'(k_c r) = -J_1(k_c r)$ where

$$k_c^2 = \varepsilon_r\left(\frac{\omega}{c}\right)^2 - k_g^2\ ,$$

see eqn.(10.90b).

The component E_θ must be zero at the waveguide wall in order that the tangential component of the electric field be zero:

$$J_1(k_c R) = 0,$$

or $$k_c R = 3.8317 \quad \text{for the lowest mode.}$$

Thus $$k_c = \frac{3.832}{0.025} = 153.3\ \text{m}^{-1}.$$

For an air-filled waveguide $\varepsilon_r = 1$, so

$$k_g^2 = 2.0373\text{x}10^4 \text{ m}^{-2} \text{ since } \frac{\omega}{c} = 209.44 \text{ m}^{-1} \text{ at 10 GHz.}$$

Consequently, $k_g = 142.7 \text{ m}^{-1}$ and the guide wavelength is $\lambda_g = \frac{2\pi}{k_g} = 4.40$ cm. But E_θ must vanish at the cavity end walls and therefore E_θ must be proportional to $\text{Sin}(\frac{n\pi z}{L})$. Thus $k_g = \frac{n\pi}{L}$ and the cavity length must be an integral number of half-wavelengths long. A convenient choice would be L= 4.40 cm.