Static vs Falling: Time slicings of Schwarzschild black holes

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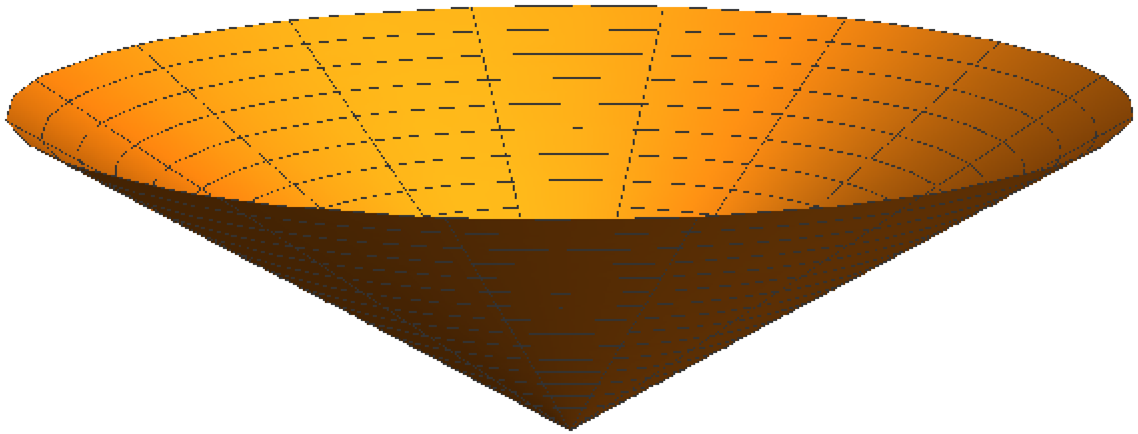
Outline

- Radial distance in Schwarzschild spacetime
  - Adapted coordinates
  - Spatial projector
  - Tetrads
  - Radar metric
- Flamm's paraboloid
- Volume
- *r* coordinate vector
- "Length-contraction"
- Bonus:
  - Coordinate choices
  - Tidal forces (elementary derivation)
  - *r* as "reduced circumference"
  - Observer at infinity

Purposes:

- Conceptual foundations
- Pedagogy
- Avoid misconceptions



Slicing by family of observers

- Parametrise radially-falling observers by *e*, dubbed "energy per unit mass":

$$e = -\mathbf{u} \cdot \xi$$

- Measurements strictly *local*

Metaphor	Energy per mass	Allowed region
"hail"	$e > 1$	all
"rain"	$e = 1$	all
"drip"	$0 < e < 1$	$r \leq \frac{2M}{1-e^2}$
"mist"	$e \leq 0$	$r < 2M$

Metric for the Rain Frame

We want a metric in the coordinates r, ϕ, and t_{rain} We make this transition in two jumps for events outside the horizon from bookkeeper coordinates to shell coordinates, then from shell coordinates to rain coordinates Assume that the resulting metric is valid inside the horizon as well as outside The transition from bookkeeper coordinates to shell coordinates is given by equations [C] and [D] in Selected Formulas at the end of this book

$$dr_{shell} = \frac{dr}{\left(1 - \frac{2M}{r}\right)^{1/2}} \qquad [D]$$

$$dt_{shell} = \left(1 - \frac{2M}{r}\right)^{1/2} dt \qquad [C]$$

Now, to go from shell coordinates to rain coordinates, use the Lorentz transformation of special relativity Choose the "rocket" coordinates to be those of the rain frame and the "laboratory" coordinates to be those of the shell frame The Lorentz transformation for differentials (page 103 of *Spacetime Physics*) is expressed for motion along the x-axis, which in this case lies along the radially inward direction

$$dt_{rain} = -v_{rel}\gamma dr_{shell} + \gamma dt_{shell} \qquad [9]$$

Substitute equations [C] and [D] into the Lorentz transformation equation [9] to obtain

$$dt_{rain} = -\frac{v_{rel}\gamma dr}{(1-2M/r)^{1/2}} + \gamma(1-2M/r)^{1/2}dt \qquad [10]$$

Solve for dt

$$dt = \frac{dt_{rain}}{\gamma(1-2M/r)^{1/2}} + \frac{v_{rel}dr}{(1-2M/r)} \qquad [11]$$

Substitute v_{rel} from equation [1] on page B-5 into the expression for the stretch factor γ

$$v_{rel} = -\left(\frac{2M}{r}\right)^{1/2} \qquad [12]$$

$$\gamma = \frac{1}{(1-v_{rel}^2)^{1/2}} = \frac{1}{\left(1 - \frac{2M}{r}\right)^{1/2}} \qquad [13]$$

Substitute equations [12] and [13] into [11] to obtain

$$dt = dt_{rain} - \frac{(2M/r)^{1/2}dr}{1 - \frac{2M}{r}} \qquad [14]$$

The Schwarzschild metric is equation [A] in the Selected Formulas at the end of this book

$$d\tau^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \frac{dr^2}{1 - \frac{2M}{r}} - r^2 d\phi^2 \qquad [A]$$

Substitute expression [14] into the Schwarzschild metric and collect terms to obtain the global rain metric in r, ϕ, t_{rain}

$$d\tau^2 = \left(1 - \frac{2M}{r}\right)dt_{rain}^2 - 2\left(\frac{2M}{r}\right)^{1/2} dt_{rain}dr - dr^2 - r^2 d\phi^2 \qquad [15]$$

This metric can be used anywhere around a nonrotating black hole, not just inside the horizon Our ability to write the metric in a form without infinities at $r = 2M$ is an indication that the plunger feels no jerk or jolt as she passes through the horizon

Lorentz boost from Schwarzschild to Gullstrand-Painleve coordinates.

Taylor & Wheeler, *Exploring Black Holes* (2000), §B4

Adapted coordinates: Lorentz boost

- Generalisation to $-\infty < e < \infty,\ e \neq 0$
- Orthonormal dual basis:

$$t' \equiv \left(1 - \frac{2M}{r}\right)^{-1/2} dt \qquad r' \equiv \left(1 - \frac{2M}{r}\right)^{1/2} dr \qquad ds^2 = -t'^2 + r'^2 + \cdots$$

- Lorentz boost to new time coordinate *T*:

$$dT \equiv \gamma(t' - V r')$$

$$u^\mu_{\text{static}} = \left(\left(1 - \frac{2M}{r}\right)^{-1/2}, 0, 0, 0\right) \qquad u^\mu_{\text{faller}} = \left(e\left(1 - \frac{2M}{r}\right)^{-1}, -\sqrt{e^2 - 1 + \frac{2M}{r}}, 0, 0\right)$$

$$\gamma = -\mathbf{u}_{\text{static}} \cdot \mathbf{u}_{\text{faller}} = e\left(1 - \frac{2M}{r}\right)^{-1/2} \qquad V = \pm\sqrt{1 - \gamma^{-2}} = -\frac{1}{|e|}\sqrt{e^2 - 1 + \frac{2M}{r}}$$

Generalised Gullstrand-Painlevé coordinates

$$ds^2 = -\frac{1}{e^2}\left(1-\frac{2M}{r}\right)dT^2+\frac{2}{e^2}\sqrt{e^2-1+\frac{2M}{r}}dT\,dr+\frac{1}{e^2}dr^2+r^2(d\theta^2+\sin^2\theta\,d\phi^2)$$

$$e \neq 0$$

- The original Gullstrand-Painleve coordinates are for "rain", *e*=1

$$ds^2 = -\left(1-\frac{2M}{r}\right)dT^2+\sqrt{\frac{2M}{r}}dT\,dr+dr^2+r^2(d\theta^2+\sin^2\theta\,d\phi^2)$$

- (Bonus: generalised Lemaitre coordinates) $e \neq 0$

$$ds^2 = -dT^2+\frac{1}{e^2}\left(e^2-1+\frac{2M}{r}\right)d\rho_e^2+r^2(d\theta^2+\sin^2\theta\,d\phi^2)$$

Coordinates history

rain ($e=1$)
Gullstrand (1922)
Painleve (1921)
Lemaitre (1932)
Robertson, etc.

drips ($0<e<1$)
Gautreau &
Hoffmann (1978)

hail & drips ($e>0$)
Martel &
Poisson (2001)

Finch (2015)

Radial distance from adapted coordinates

Set: $$dT = d\theta = d\phi = 0$$

$$dL = \frac{1}{|e|} dr$$

Gautreau & Hoffman (1978), 0<e<1

$$L = \int_{R_1}^{R_2} \frac{dR}{(1 - 2M/R_i)^{1/2}} = \frac{R_2 - R_1}{(1 - 2M/R_i)^{1/2}}$$

$$e = \left(1 - \frac{2M}{R_i}\right)^{-1/2}$$

Spatial projector

- Given some metric **g** and observer 4-velocity **u**, the spatial projector tensor is:

$$P_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$$

- Spatial distance measured by this observer is:

$$dL^2 = P_{\mu\nu} dx^\mu dx^\nu$$

- In direction *r*, radial distance is:

$$dL = \sqrt{P_{rr}} dr$$

Spatial projector: Schwarzschild coordinates

$$P_{\mu\nu} = \begin{pmatrix} e^2 - 1 + \frac{2M}{r} & -e\sqrt{e^2 - 1 + \frac{2M}{r}}\left(1 - \frac{2M}{r}\right)^{-1} & 0 & 0 \\ -e\sqrt{e^2 - 1 + \frac{2M}{r}}\left(1 - \frac{2M}{r}\right)^{-1} & e^2\left(1 - \frac{2M}{r}\right)^{-2} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

$$dL = e\left(1 - \frac{2M}{r}\right)^{-1} dr$$

Spatial projector: generalised Gullstrand-Painleve coordinates

$$P_{\mu\nu} = \begin{pmatrix} \frac{1}{e^2}(e^2 - 1 + \frac{2M}{r}) & -\frac{1}{e^2}\sqrt{e^2 - 1 + \frac{2M}{r}} & 0 & 0 \\ -\frac{1}{e^2}\sqrt{e^2 - 1 + \frac{2M}{r}} & \frac{1}{e^2} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

$$dL = \frac{1}{|e|} dr$$

- Same tensor **P**, same coordinate *r*, but different result! (Due to different time slicings, in fact the r-coordinate vectors are different…)
- The 2 formulae concur for static observers, for whom: $e = \sqrt{1 - \frac{2M}{r}}$
- Static observers measure the usual distance: $dL = \left(1 - \frac{2M}{r}\right)^{-1/2} dr$

Observer frame: orthonormal tetrad $\{\mathbf{e}_{\hat{\alpha}}\}$

- Orthonormal ➔ 1 timelike, 3 spacelike vectors

$$_{(\text{static})}e^{\mu}_{\hat{0}} = \left(\left(1-\frac{2M}{r}\right)^{-1/2}, 0, 0, 0\right)$$

$$_{(\text{static})}e^{\mu}_{\hat{1}} = \left(0, \left(1-\frac{2M}{r}\right)^{1/2}, 0, 0\right)$$

$$_{(\text{static})}e^{\mu}_{\hat{2}} = \left(0, 0, \frac{1}{r}, 0\right)$$

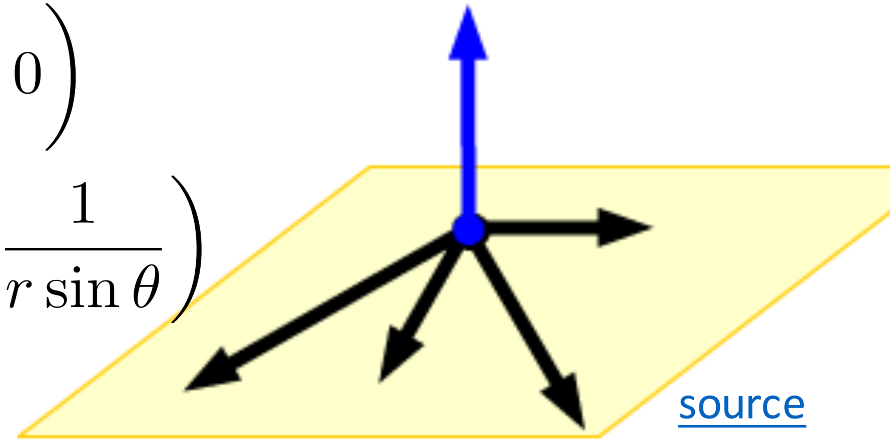
$$_{(\text{static})}e^{\mu}_{\hat{3}} = \left(0, 0, 0, \frac{1}{r\sin\theta}\right)$$

$$_{(\text{faller})}e^{\mu}_{\hat{0}} = \left(e\left(1-\frac{2M}{r}\right)^{-1}, -\sqrt{e^2-1+\frac{2M}{r}}, 0, 0\right)$$

$$_{(\text{faller})}e^{\mu}_{\hat{1}} = \left(-\left(1-\frac{2M}{r}\right)^{-1}\sqrt{e^2-1+\frac{2M}{r}}, e, 0, 0\right)$$

$$_{(\text{faller})}e^{\mu}_{\hat{2}} = \left(0, 0, \frac{1}{r}, 0\right)$$

$$_{(\text{faller})}e^{\mu}_{\hat{3}} = \left(0, 0, 0, \frac{1}{r\sin\theta}\right)$$



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Tetrad decomposition

$$\left({}_{\text{(static)}}\partial_r\right)^\mu = (0,1,0,0)$$ in Schwarzschild coordinates

$$\left({}_{\text{(faller)}}\partial_r\right)^\mu = (0,1,0,0)$$ in generalised Gullstrand-Painleve coordinates

$$= \left(-\frac{1}{e}\left(1-\frac{2M}{r}\right)^{-1}\sqrt{e^2-1+\frac{2M}{r}}, 1, 0, 0\right)$$ Schwarzschild coordinates

$${}_{\text{(static)}}\partial_r = -\left(1-\frac{2M}{r}\right)^{-1}\sqrt{e^2-1+\frac{2M}{r}}\;{}_{\text{(faller)}}\mathbf{e}_{\hat{0}} + e\left(1-\frac{2M}{r}\right)^{-1}\;{}_{\text{(faller)}}\mathbf{e}_{\hat{1}}$$

$${}_{\text{(static)}}\partial_r = \left(1-\frac{2M}{r}\right)^{-1/2}\;{}_{\text{(static)}}\mathbf{e}_{\hat{1}}$$

$${}_{\text{(faller)}}\partial_r = \frac{1}{e}\;{}_{\text{(faller)}}\mathbf{e}_{\hat{1}}$$

$${}_{\text{(faller)}}\partial_r = \frac{1}{e}\left(1-\frac{2M}{r}\right)^{-1/2}\sqrt{e^2-1+\frac{2M}{r}}\;{}_{\text{(static)}}\mathbf{e}_{\hat{0}} + \left(1-\frac{2M}{r}\right)^{-1/2}\;{}_{\text{(static)}}\mathbf{e}_{\hat{1}}$$

Radar metric

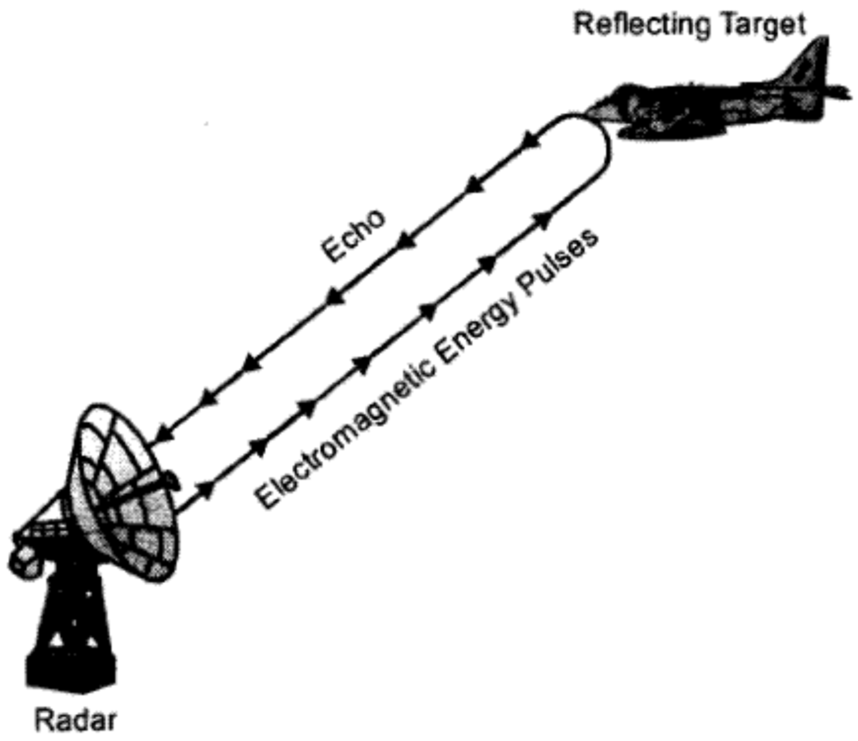
$$L = \frac{1}{2} c\tau$$

$$\gamma_{ij} \equiv g_{ij} - \frac{g_{0i} g_{0j}}{g_{00}} \qquad (i, j = 1, 2, 3)$$

$$dL^2 = \gamma_{ij} dx^i dx^j$$

(Landau & Lifshitz 1971, §84)

Observers with the radar gun must be comoving. But the spatial projector reduces to this same form when **u** is comoving! Hence it is the same as the radar metric.



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Textbooks on Schwarzschild *r*:

- Of *r*: "It is *not* the distance from any 'center.'" (Hartle §9.1)
- Of $\left(1 - \frac{2M}{r}\right)^{-1/2} dr$

  "radial ruler distance" (Rindler 2006, p230)
  "physical distance", "actual radial distance" (Moore 2012, p106-7)
- Newtonian: a coordinate is either distance or it is not, and this is the same for all observers
- GR: a coordinate is not the distance, nor is it *not* the distance. Rather, it depends on who is measuring

Space

- 3-dimensional "space" part of spacetime
- In Schw. coords for *r*>2*M*, take *dt*=0 and equatorial slice *dφ*=0:

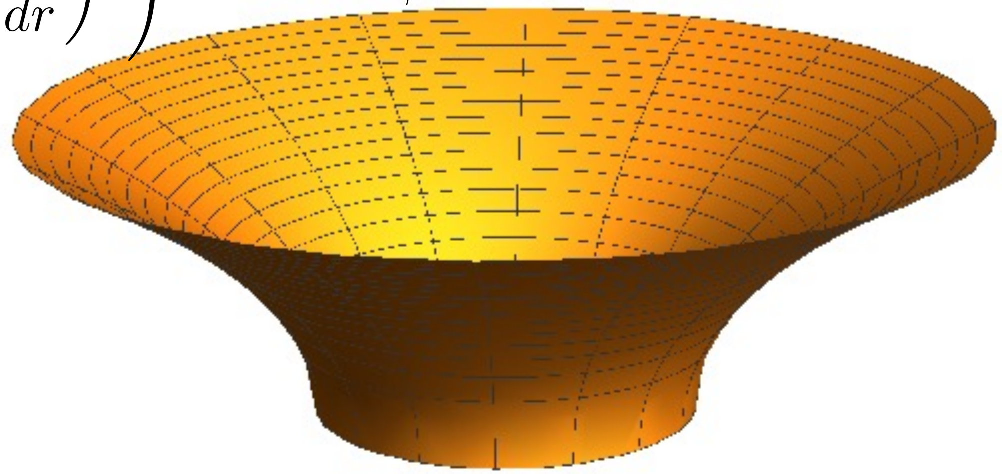
$$ds^2 = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\phi^2$$

- Represent as a 2-dimensional surface *z*=*z*(*r*) in Euclidean space, with the same curvature:

$$ds^2 = dx^2 + dy^2 + dz^2 = \left(1 + \left(\frac{dz}{dr}\right)^2\right) dr^2 + r^2 d\phi^2$$

- Flamm's paraboloid (1916):

$$z = \pm 2\sqrt{2M(r - 2M)}$$



Space

- In generalised G-P coordinates, set $dT = d\varphi = 0$:

$$ds^2 = \frac{1}{e^2}dr^2 + r^2 d\phi^2$$

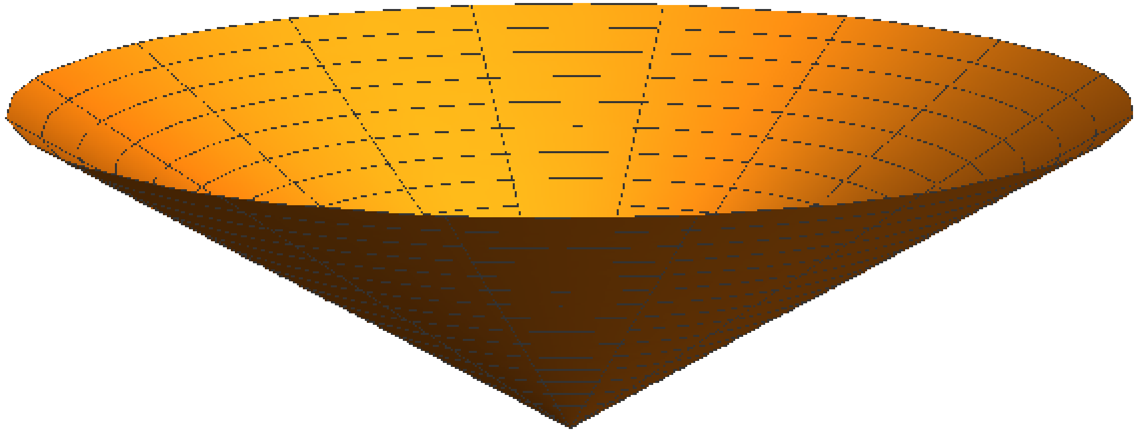
- Then

$$ds^2 = dx^2 + dy^2 + dz^2 = \left(1 + \left(\frac{dz}{dr}\right)^2\right) dr^2 + r^2 d\phi^2$$

- A cone for $|e|<1$. $|e|=1$ gives a flat plane. $|e|>1$ cannot be represented by this method.

$$z = \pm\sqrt{\frac{1}{e^2} - 1}\, r \ \propto \ r$$

- Moore: space cannot be represented inside the horizon



Volume

- Spatial 3-volume
- But what "space" is depends on the observer
- Volume inside r=2M is closely related to the volume of a Euclidean ball of radius r=2M. Finch gives this for e>0

$$V = \frac{1}{|e|}\frac{4\pi(2M)^3}{3}$$

- More rigorously:

$$\sqrt{|g_3|}d^3x = \frac{1}{|e|}\sqrt{e^2 - 1 + \frac{2M}{r}}r^2 \sin\theta \, d\rho \wedge d\theta \wedge d\phi$$

Time and space inside the event horizon

- Do time and space swap roles?
  - “Space and time themselves do not interchange roles. Coordinates do…” (Taylor & Wheeler §3.7)
  - “The most obvious pathology at $r = 2M$ is the reversal there of the roles of t and r as timelike and spacelike” (MTW §31.3)
  - “First, the coordinate r for $r < r\_g$ plays the role of a time coordinate, and t acts as a spatial coordinate.” (Frolov & Novikov §14.2)

Time and space inside the event horizon

- A coordinate is timelike/null/spacelike depending on its *coordinate vector*
- The *r*-coordinate vector points in the direction of increasing *r*:

$$(\partial_r)^\mu = (0,1,0,0)$$

- This vector has norm-squared:

$$\partial_r \cdot \partial_r = g_{\mu\nu}(\partial_r)^\mu(\partial_r)^\nu = g_{rr}$$

- For Schwarzschild coordinates, this is spacelike (negative in our convention) for $r<2M$

$$g_{rr} = \left(1 - \frac{2M}{r}\right)^{-1} < 0$$

- But for generalised Gullstrand-Painlevé coordinates, it is spacelike everywhere

$$g_{rr} = \frac{1}{e^2} > 0$$

Time and space inside the event horizon

All these coordinate systems use the same $r : \mathcal{M} \to \mathbb{R}$

Coordinates	Inner product $\partial_r \cdot \partial_r = g_{rr}$	Interpretation
Schwarzschild	$\left(1 - \frac{2M}{r}\right)^{-1}$	spacelike for $r > 2M$ timelike for $r < 2M$
generalised Gullstrand-Painlevé	$\frac{1}{e^2}$	spacelike
generalised Lemaître	$\frac{1}{e^2}(e^2 - 1 + \frac{2M}{r})$	spacelike
Eddington-Finkelstein (null version)	0	null
Eddington-Finkelstein (timelike version)	$1 + \frac{2M}{r}$	spacelike

Time coordinate

- Unlike for *r* the "time" coordinates are distinct. But they are not always timelike:

Coordinates	Interval $\partial_T \cdot \partial_T = g_{TT}$	Interpretation
Schwarzschild	$-\left(1 - \frac{2M}{r}\right)^{-1}$	timelike for $r > 2M$ spacelike for $r < 2M$
Eddington-Finkelstein (null version)	$-\left(1 - \frac{2M}{r}\right)$	ditto, but added $r = 2M$ where it is null
Eddington-Finkelstein (timelike version)	$-\left(1 - \frac{2M}{r}\right)$	ditto
generalised Gullstrand-Painlevé	$-\frac{1}{e^2}\left(1 - \frac{2M}{r}\right)$	ditto
generalised Lemaître	-1	timelike everywhere

Important implications: they are *different* vectors!

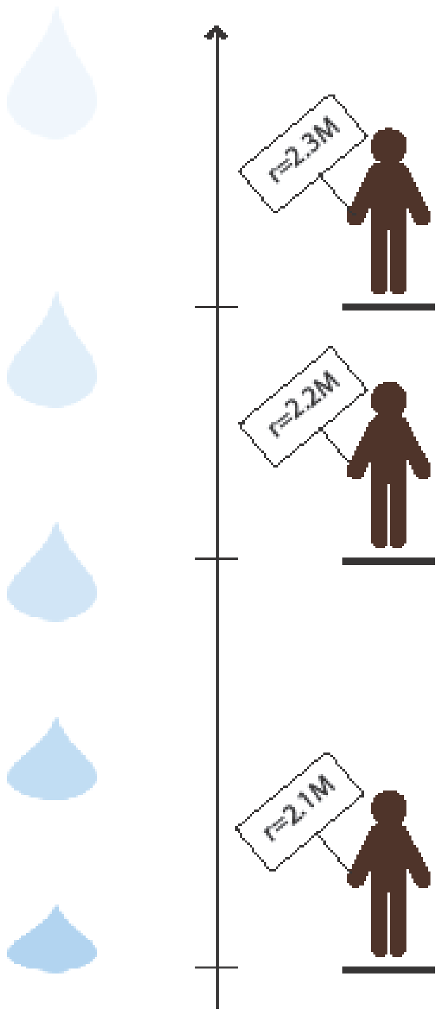
Time and space inside the event horizon

- What causes decrease in *r*?
  - "… you cannot even stop yourself from moving in the direction of decreasing *r*, since this is simply the timelike direction. …*t* becomes spacelike and *r* becomes timelike. Thus you can no more stop moving toward the singularity than you can stop getting older." (Carroll §5.7)
  - "The *r* = 0 singularity in the Schwarzschild geometry is not a place in space; it is a moment in time." (Hartle §12.1)

Time and space inside the event horizon

- Hypersurface *r=const* has normal vector: $$\xi^\mu = g^{r\mu}$$
- This vector has norm-squared: $$g^{rr} = 1 - \frac{2M}{r} < 0$$
- The normal vector is timelike, so the hypersurface is spacelike
- This holds for all coordinate systems using *r*
- So coordinate vectors depend on the other coordinates,
  but constant coordinate surfaces do not.

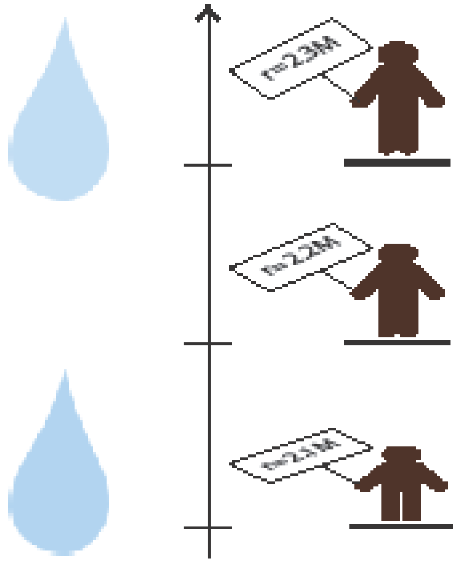
Radial distance: length-contraction



$$\gamma = -\mathbf{u}_{\text{static}} \cdot \mathbf{u}_{\text{faller}} = e\left(1 - \frac{2M}{r}\right)^{-1/2}$$

$$\left(1 - \frac{2M}{r}\right)^{-1/2} \frac{1}{|e|}\left(1 - \frac{2M}{r}\right)^{1/2} dr = \frac{1}{|e|} dr$$

$$\left(1 - \frac{2M}{r}\right)^{-1/2} e\left(1 - \frac{2M}{r}\right)^{-1/2} dr = e\left(1 - \frac{2M}{r}\right)^{-1} dr$$



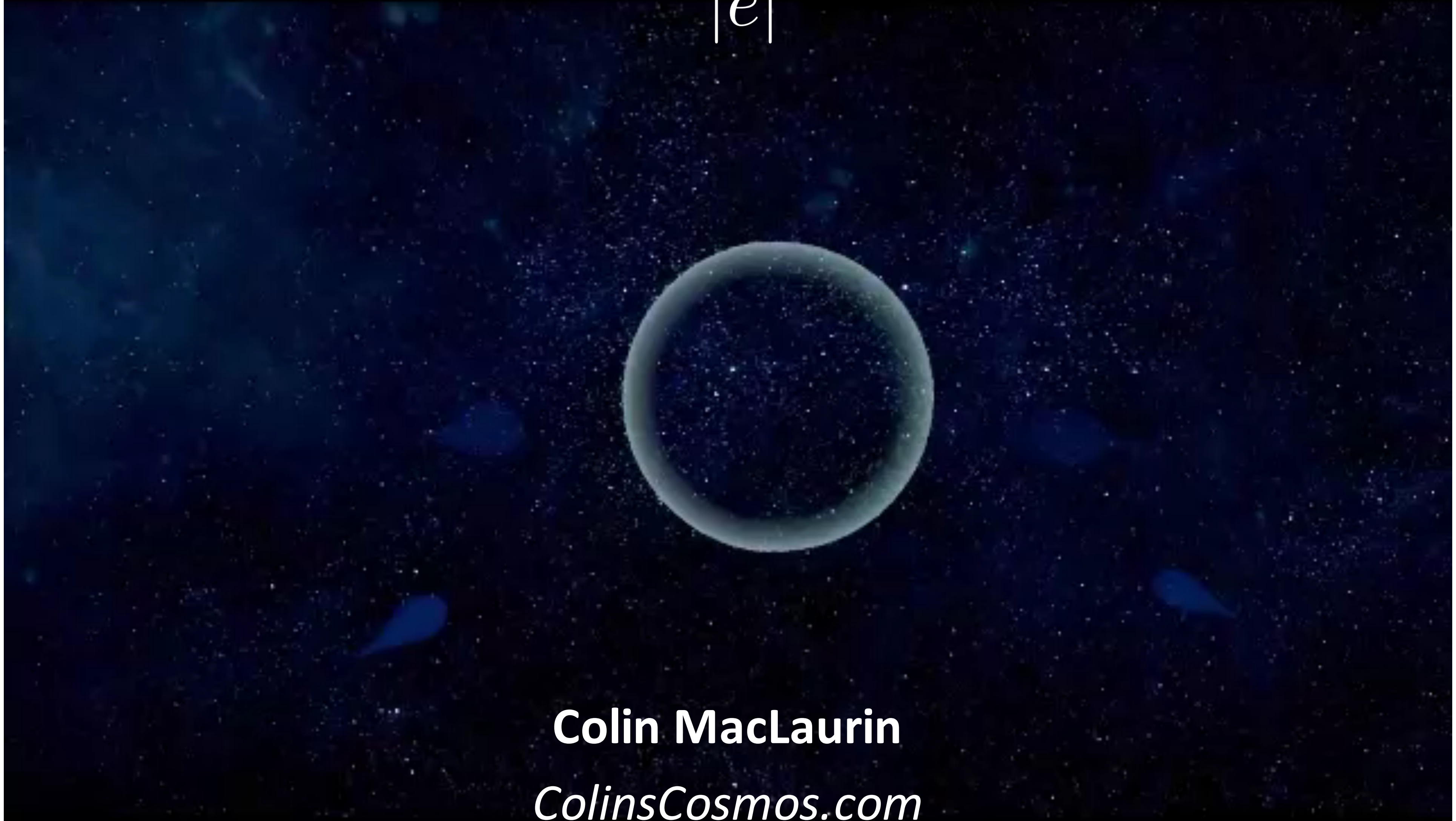
Conclusion

- Emphasised relativity of space, time, and simultaneity
- Avoids over-interpretation of Schwarzschild coordinates. Eisenstaedt described this “neo-Newtonian” interpretation reigning until the 1960s “renaissance”. However vestiges remain. As do misconceptions.

References

- C. MacLaurin, T. Davis, G. Lewis (2016), in preparation
- Taylor & Wheeler, *Exploring Black Holes* (2000)
- Gautreau & Hoffman, "The Schwarzschild radial coordinate as a measure of proper distance" (1978)
- Martel & Poisson, "Regular coordinate systems for Schwarzschild and other spherical spacetimes" (2001)
- Finch, "Coordinate families for the Schwarzschild geometry based on radial timelike geodesics" (2015)

Main formula: $dL = \frac{1}{|e|}dr$ (& gratuitous eye candy)



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Adapted coordinates

	Best suited to	Comoving?	Comments
Eddington-Finkelstein	photons	yes	
Kruskal-Szekeres	photons	no (but "lightcone" variant is)	maximal analytic extension
Generalised Gullstrand-Painleve	massive particles: e-faller	no	
Generalised Lemaitre	massive particles: e-faller	yes	
Schwarzschild	massive particles: static (r>2M) e=0 (r<2M)	yes	

All except Schwarzschild are non-singular at $r=2M$

Bonus: elementary derivation of tidal forces

- All radially moving observers experience the same tidal forces: "amazing result (a consequence of special algebraic properties of the Schwarzschild geometry…)" (MTW §31.2, 32.6)
- Usually lots of tensor algebra. But the length-contraction result gives an "elementary" proof. This generalises Taylor & Wheeler (§B.7) *e*=1

$$\frac{dr}{d\tau} = -\sqrt{e^2 - 1 + \frac{2M}{r}} \qquad \frac{dL}{d\tau} = -\frac{1}{|e|}\sqrt{e^2 - 1 + \frac{2M}{r}}$$

$$g \equiv \frac{d^2L}{d\tau^2} = -\frac{M}{|e|r^2} \qquad \frac{dg}{dL} = \frac{2M}{r^3}$$

Bonus: *r* as "reduced circumference"

- Draw a circle, circumference 2 pi r, so can recover r (Droste, 1910s)
- Only for static observers! (Or radial motion, if could measure fast enough)
- Compare the rotating disk in Minkowski spacetime, aka Ehrenfest's Paradox. This has a non-Euclidean geometry, with circumference not 2 pi r

Bonus: *t* time of observer at infinity

- t as time of observer at infinity.
- Con: simultaneity is relative. Especially at such great separations
- E.g. raindrop time (Gullstrand-Painleve coordinates). Then *raindrop* time is the time at infinity! (How much time to fall past the event horizon? Distant observer would answer: exactly the same time as the proper time of the raindrop itself!)
- Pro: extend spatial r-coordinate inwards, and indeed the 4-velocities line up
- Mixed: Riemann normal coordinates. Pro as above. But con if slicing by the induced time coordinate, the geodesic extension of the timelike 4-velocity… but we know what this is; it is raindrop time!
- Con: Fermi coordinates. Consider observer freefalling
- Better comparison: static observers (at all $r>2M$). Better, but not exact. The spacetime slicings are the same. While *t* is not their proper time, it is proportional to it (since constant r).

- Lemaître (1932):
    - We show that the singularity of the Schwarzschild exterior is an apparent singularity due to the fact that one has imposed a static solution and that it can be eliminated by a change of coordinates.