A Cursory Introduction to General Relativity

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Outline

A Little History

- At the turn of the 20th century, the laws of electrodynamics and mechanics contradicted each other.
- Galiean mechanics contained no reference to the speed of light, but Maxwells equations and experiments said that light goes at the speed of light no matter how fast you are going.
- To deal with this people argued that there should be new rules to add velocities and that the results of measuring an objects mass or length as it approaches the speed of light would defy ones expectations.

Enter Einstein (1)

- Einstein argued that the constancy of the speed of light was a property of space(time) itself.
- The Newtonian picture was that everyone shared the same view of space and time marched in lockstep for everybody.
- So people would agree on the length of objects and the duration of time between events

$$dl^2 = dx^2 + dy^2 + dz^2, dt$$

.

Enter Einstein (2)

- Einstein argued that spacetime was the important concept and that the interval between events was what everyone could agree on.

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

.

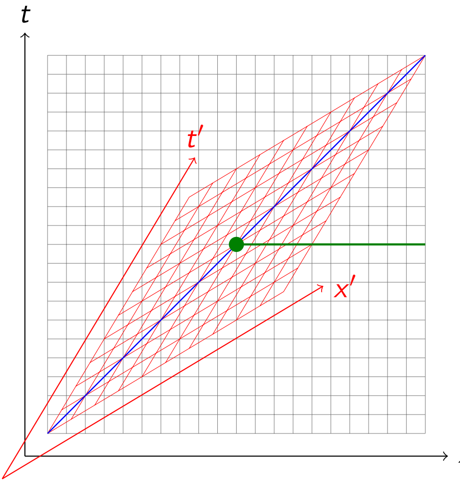
- This simple idea explained all of the nuttiness that experiments with light uncovered but it also cast the die for the downfall of Newtonian gravity.

Exit Newton (1)

- Newtonian gravity was action at a distance (Newton himself wasnt happy about this). This means that if you move a mass, its gravitational field will change everywhere instantaneously.
- In special relativity this leads to contradictions.

Exit Newton (2)

- At the event marked by the circle, a mass is shaken, the gravitational field will change instantly along green line.
- Someone moving relative to the mass will find that the field changes before the mass is moved.
- This is bad, bad, bad.



Einstein Again

- Instead of trashing the brand-new special theory of relativity, Einstein decided to rework the venerable theory of gravity. He came upon the general theory of relativity.
- Newtonian gravity looks a lot like electrostatics:

$$\nabla^2 \phi = 4\pi G \rho$$

.

- Lets generalize it as a relativistic scalar field:

$$\nabla^2 \phi - c^2 \frac{d^2 \phi}{dt^2} = 4\pi G \rho$$

.

- What is ρ? The mass (or energy) density that one measures depends on velocity but the L.H.S. does not, so this equation is not Lorentz invariant.

A First Try (1)

- The relativistic generalization of the mass or energy density is the energy-momentum tensor. For a perfect fluid you have,

$$T^{\alpha\beta} = \left(\rho + \frac{p}{c^2}\right) u^\alpha u^\beta + p g^{\alpha\beta}$$

where

$$g^{\alpha\beta} = \mathrm{diag}\,(1, -1, -1, 1)$$

is the metric tensor.

- We can get a scalar by taking

$$T = g_{\alpha\beta} T^{\alpha\beta} = \rho c^2 - 3p$$

so

$$\nabla^2 \phi - c^2 \frac{d^2\phi}{dt^2} = 4\pi \frac{G}{c^2} T$$

.

A First Try (2)

- The energy-momentum tensor of an electromagnetic field is traceless so $T = 0$.
- This means that photons or the energy in a electric field does not generate gravity.
- This is bad, bad, bad.
- If photons feel gravity, momentum is not conserved.
- If photons dont feel gravity, energy is not conserved.

What to do?

- The next obvious step would be a vector field like electromagnetism, but it isnt obvious how to make a vector from the energy-momentum tensor.
- How about a tensor field? So

$$\Box^2 h^{\alpha\beta} = 4\pi \frac{G}{c^2} T^{\alpha\beta}$$

- But we would like gravitational energy to gravitate, so h should be on both sides.
- One can develop a theory equivalent to GR like this.

Einstein's Solution

- Einstein assumed that all objects follow the same paths in a gravitational field regardless of their mass or internal composition (strong equivalence principle),
- He suggested that gravity is the curvature of spacetime.
- Objects follow extremal paths in the spacetime (geodesics).
- Therefore, the metric itself ($g_{\alpha\beta}$) contains the hallmarks of gravity.

The Geodesic Equation (1)

- Let's make some definitions:

$$u^\alpha = \frac{dx^\alpha}{ds}, u_\alpha = g_{\alpha\beta} u^\beta, g_{\alpha\beta,\gamma} = \frac{dg_{\alpha\beta}}{dx^\gamma}$$

- Using the definition of the metric

$$\begin{aligned} \delta\left(ds^2\right) &= 2ds\delta\left(ds\right) = \delta\left(g_{\alpha\beta} dx^\alpha dx^\beta\right) \\ &= dx^\alpha dx^\beta g_{\alpha\beta,\gamma}\delta x^\gamma + 2g_{\alpha\beta} dx^\alpha d\left(\delta x^\beta\right) \end{aligned}$$

- Solving for $\delta(ds)$ and integrating by parts yields

$$\begin{aligned} \delta s &= \int \delta(ds) = \int \left[\frac{1}{2} u^\alpha u^\beta g_{\alpha\beta,\gamma}\delta x^\gamma + g_{\alpha\gamma} u^\alpha \frac{d\delta x^\gamma}{ds}\right] ds \\ &= \int \left[\frac{1}{2} u^\alpha u^\beta g_{\alpha\beta,\gamma}\delta x^\gamma - \frac{d}{ds}\left(g_{\alpha\gamma} u^\alpha\right)\delta x^\gamma\right] ds \end{aligned}$$

The Geodesic Equation (2)

- Because the variation is arbitrary we can set its coefficient equal to zero

$$\frac{du_\gamma}{ds} - \frac{1}{2} u^\alpha u^\beta g_{\alpha\beta,\gamma}$$

$$\begin{aligned}
\frac{1}{2} u^\alpha u^\beta g_{\alpha\beta,\gamma} - \frac{d}{ds}\left(g_{\alpha\gamma} u^\alpha\right) &= 0 \\
\frac{1}{2} u^\alpha u^\beta g_{\alpha\beta,\gamma} - g_{\alpha\gamma}\frac{du^\alpha}{ds} - u^\alpha u^\beta g_{\alpha\gamma,\beta} &= 0 \\
g_{\alpha\gamma}\frac{du^\alpha}{ds} + \frac{1}{2} u^\alpha u^\beta \left(g_{\gamma\alpha,\beta} + g_{\gamma\beta,\alpha} - g_{\alpha\beta,\gamma}\right) &= 0
\end{aligned}$$

The Geodesic Equation (3)

- After rearranging we can write

$$g_{\alpha\gamma}\frac{du^\alpha}{ds} + \Gamma_{\gamma,\alpha\beta}u^\alpha u^\beta = 0$$

where the connection coefficient or Christoffel symbol is given by

$$\Gamma_{\gamma,\alpha\beta} = \frac{1}{2}\left(g_{\gamma\alpha,\beta} + g_{\gamma\beta,\alpha} - g_{\alpha\beta,\gamma}\right).$$

Tensors

- The quantities like $T_{\alpha\beta}$ that we have been manipulating are called tensors, and they have special properties.
- Specifically they transform simply under coordinate transformations.

$$T_{\alpha\beta} = \frac{\partial x'^\gamma}{\partial x^\alpha} \frac{\partial x'^\delta}{\partial x^\beta} T'_{\gamma\delta}, \; T^{\alpha\beta} = \frac{\partial x^\alpha}{\partial x'^\gamma} \frac{\partial x^\beta}{\partial x'^\delta} T'^{\gamma\delta},$$

- Also if metric isn't constant you would expect derivatives to depend on how the coordinates change as you move too.

Tensors (2)

- We want a derivative that transforms like a tensor (this is also called the connection).
- The derivative of a scalar quality should be simple; it does not refer to any directions, so we define the covariant derivative to be $\phi_{;\alpha} = \phi_{,\alpha}$.
- Let's assume that the chain and product rules work for the covariant derivative like the normal one that we are familiar with (also linearity).
- Let's prove a result about the metric, the tensor that raises and lowers indices.

$$A_{\beta;\alpha} = \left(g_{\beta\gamma}A^{\gamma}\right)_{;\alpha} = g_{\beta\gamma;\alpha}A^{\gamma} + g_{\beta\gamma}A^{\gamma}_{;\alpha} = g_{\beta\gamma;\alpha}A^{\gamma} + A_{\beta;\alpha} =$$

so because the vector field A_{β} is arbitrary, $g_{\beta\gamma;\alpha} = 0$.

Tensors (3)

- Let's define the covariant derivative of a tensor that is compatible with our requirements.

$$A_{\alpha\beta;\gamma} = A_{\alpha\beta,\gamma} - A_{\delta\beta}\Gamma^{\delta}{}_{\alpha\gamma} - A_{\alpha\delta}\Gamma^{\delta}{}_{\beta\gamma}.$$

- Let's apply this to the metric itself to get

$$g_{\alpha\beta;\gamma} = g_{\alpha\beta,\gamma} - g_{\delta\beta}\Gamma^{\delta}{}_{\alpha\gamma} - g_{\alpha\delta}\Gamma^{\delta}{}_{\beta\gamma}.$$

- The left-hand side is zero. Furthermore, if we also assume that the derivative is symmetric (torsion-free), then $\Gamma^{\delta}{}_{\beta\gamma}$ is symmetric in its lower indices.

Tensors (4)

- We can use the symmetry of the metric and the connection to get the following three relations.

$$
\begin{aligned}
0 &= g_{\alpha\beta,\gamma} - g_{\delta\beta}\Gamma^{\delta}{}_{\alpha\gamma} - g_{\alpha\delta}\Gamma^{\delta}{}_{\beta\gamma} \\
0 &= g_{\gamma\beta,\alpha} - g_{\delta\beta}\Gamma^{\delta}{}_{\alpha\gamma} - g_{\gamma\delta}\Gamma^{\delta}{}_{\beta\alpha} \\
0 &= g_{\alpha\gamma,\beta} - g_{\delta\gamma}\Gamma^{\delta}{}_{\alpha\beta} - g_{\alpha\delta}\Gamma^{\delta}{}_{\beta\gamma}
\end{aligned}
$$

- To get the second expression, we swapped α and γ. To get the third expression, we swapped β and γ.
- Now let's add the first two and subtract the third, cancelling terms that are equal by symmetry.

$$
0 = g_{\alpha\beta,\gamma} + g_{\gamma\beta,\alpha} - g_{\alpha\gamma,\beta} - 2g_{\delta\beta}\Gamma^{\delta}{}_{\alpha\gamma}
$$

$$
\Gamma^{\delta}{}_{\alpha\gamma} = \frac{1}{2}g^{\delta\beta}\left(g_{\alpha\beta,\gamma} + g_{\gamma\beta,\alpha} - g_{\alpha\gamma,\beta}\right)
$$

The Derivative of a Tensor

- How the components of a tensor vary reflect both the change in the coordinates and the physical variance.
- We can remove the coordinate part and focus on the physics.

$$\begin{aligned} A^{\alpha}_{;\beta} &= A^{\alpha}_{,\beta} + \Gamma^{\alpha}{}_{\gamma\beta} A^{\gamma} \\ A_{\alpha;\beta} &= A_{\alpha,\beta} - \Gamma^{\gamma}{}_{\alpha\beta} A_{\gamma} \\ A^{\alpha\beta}_{;\gamma} &= A^{\alpha\beta}_{,\gamma} + \Gamma^{\alpha}{}_{\delta\gamma} A^{\delta\beta} + \Gamma^{\beta}{}_{\delta\gamma} A^{\alpha\delta} \end{aligned}$$

where

$$\Gamma^{\delta}{}_{\alpha\beta} = \frac{1}{2} g^{\delta\gamma} \left(g_{\gamma\alpha,\beta} + g_{\gamma\beta,\alpha} - g_{\alpha\beta,\gamma} \right)$$

Some Important Tensors (1)

- First, we measure scalar quantities the length of one vector along the direction of another. These scalars do not depend on the coordinate system.
- Coordinate vectors - dx^α
- Four velocity and four momentum - u^α and p^α.
- Killing vectors (ϵ^α) hold the key to the symmetry of the spacetime. The value of $\epsilon^\alpha p_\alpha$ is constant along a geodesic.
- I am moving with four-velocity u^α and I detect a particle with four-momentum p^α. I would measure an energy of $g_{\alpha\beta} u^\alpha p^\beta$.
- Of course, $g_{\alpha\beta}$ is the most important tensor of all. Without it we could not construct scalars and measure anything.

Christoffels

- The Christoffel symbols are not a tensor.
- From the rule for tensor transformation if a tensor is zero in one coordinate system it will be zero in all others.
- If I use the geodesics themselves, I can set up a coordinate system *localllly* in which the Christoffels vanish.
- However, if the geodesics diverge the Christoffels won't be zero everywhere. The separation (v^μ) of two nearby geodesics evolves as

$$\frac{d^2 v^\mu}{ds^2} = u^\nu u^\alpha v^\beta R^\mu{}_{\nu\alpha\beta}$$

$$R^\mu{}_{\nu\alpha\beta} = \Gamma^\mu{}_{\nu\beta,\alpha} + \Gamma^\mu{}_{\nu\alpha,\beta} + \Gamma^\mu{}_{\sigma\alpha}\Gamma^\sigma{}_{\nu\beta} + \Gamma^\mu{}_{\sigma\beta}\Gamma^\sigma{}_{\nu\alpha}$$

Poisson Equation to Einstein Equation

- We can finally make contact through the Riemman tensor $(R^{\mu}{}_{\nu\alpha\beta})$ to the source of gravity. According to Newton, the paths of two nearby objects diverge due to gravity as

$$\frac{d^2 v^\mu}{ds^2} = v^\beta f_{\mu,\beta} = -v^\nu \phi_{,\mu,\beta}.$$

  so $\phi_{,\beta,\beta} = 4\pi G\rho$ is related to the Ricci tensor, $R_{\nu\alpha} = R^{\beta}{}_{\nu\alpha\beta}$.
- We can write the following equation with $R = R^\alpha_\alpha$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

  where the first two terms comprise the Einstein tensor $G_{\mu\nu}$. It is important to note that $G_{\mu\nu;\nu} = 0$.

How to solve the equation?

- For an analytic solution, you need a high degree of symmetry and a simple expression for the energy-momemtum tensor:
  - Static spherically symmetric: vacuum, perfect fluid, electric field, scalar field
  - Homogeneous: perfect fluid
  - Stationary Axisymmetric: vacuum, electric field
- Numerical solutions are also difficult because the equations are non-linear.

Spherically symmetric vacuum (1)

- Let's try to find a spherically symmetric solution without matter. It starts with a trial metric:

$$ds^2 = e^{\nu} c^2 dt^2 - r^2 (d\theta^2 + sin^2\theta d\phi^2) - e^{\lambda} dr^2$$

- This equation means $ds^2 = g_{\alpha\beta} dx^{\alpha} dx^{\beta}$ so it is a compact way to write out the metric components.
- The functions ν and λ depend on t and r.
- There could also be a term proportional to $drdt$, but we can eliminated by a coordinate transformation.

Spherically symmetric vacuum (2)

- With Maple we can quickly get the non-zero components of the Einstein tensor and apply the Einstein equation:

$$\begin{aligned}
\frac{8\pi G}{c^4} T_0^0 &= -e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) + \frac{1}{r^2} \\
\frac{8\pi G}{c^4} T_1^0 &= -e^{-\lambda}\frac{\dot{\lambda}}{r} \\
\frac{8\pi G}{c^4} T_1^1 &= -e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2} \\
\frac{8\pi G}{c^4} T_2^2 &= \frac{8\pi G}{c^4} T_3^3 = \mathcal{UGLY}
\end{aligned}$$

where we have used the prime for differentiation with respect to radius and the dot for time.

Spherically symmetric vacuum (3)

- The left-hand sides equal zero, so the second equation tells us λ is just a function of radius.
- The difference of the first and third equations tell us that, $\lambda' + \nu' = 0$ so $\lambda + \nu = f(t)$. We can redefine the time coordinate such that $f(t) = 0$ so the metric is static (it is not a function of time).
- And the first equation yields

$$e^{-\lambda} = e^{\nu} = 1 + \frac{\text{constant}}{r}.$$

- We can find the value of the constant by insisting that Newtonian gravity hold as $r \to \infty$. It is $-2GM/c^2$.

Geodesics around stars (1)

- Many of the tests of the general relativity involve the motion of objects near spherically symmetric bodies where the metric is given by (taking $G = c = 1$)

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 \cos^2\theta d\phi^2 - r^2 d\theta^2.$$

- The metric does not depend on time or the angle ϕ so along a geodesic the values of u_t and u_ϕ are constant (dt and $d\phi$ are Killing vectors).
- We would also like to understand massless particles for which the four-velocity does not make sense, so we will use the four-momentum, so we have $E = p_t$ and $L = p_\phi$ as constants of the motion.

Geodesics around stars (2)

- Let's calculate p^2 using the metric to give

$$m^2 = E^2 \left(1 - \frac{2M}{r}\right)^{-1} - (p^r)^2 \left(1 - \frac{2M}{r}\right)^{-1} - \frac{L^2}{r^2}$$

- And solve for p^r in terms of the constants of motion

$$(p^r)^2 = E^2 - \left(m^2 + \frac{L^2}{r^2}\right)\left(1 - \frac{2M}{r}\right).$$

- Furthermore, we know that $\left(p^\phi\right)^2 = \frac{L^2}{r^4}$
- Combining these results yields,

$$\left(\frac{d\phi}{dr}\right)^2 = \frac{1}{r^2}\left[\frac{r^2}{b^2} - 1 + \frac{2M}{r}\left(1 + \frac{m^2 r^2}{L^2}\right)\right]^{-1}$$

where $b^2 = L^2/(E^2 - m^2)$.

Geodesics around stars (3)

- The quantity in the brackets will vanish at the turning points of the path, the minimum radius (and the maximum radius if there is one).
- We can write b^2 in terms of the minimum radius as

$$\frac{1}{b^2} = \frac{1}{r_0^2} - \frac{2M}{r_0}\left(\frac{1}{r_0^2} + \frac{m^2}{L^2}\right).$$

- This yields an equation for the orbit in terms of the angular momentum and the distance of closest approach.

$$\left(\frac{d\phi}{dr}\right)^2 = \frac{1}{r^2}\left[\frac{r^2}{r_0^2} - 1 - \frac{2Mr^2}{r_0}\left(\frac{1}{r_0^2} + \frac{m^2}{L^2}\right) + \frac{2M}{r}\left(1 + \frac{m^2 r^2}{L^2}\right)\right]^{-1}$$

- Homework: Find the angle of deflection of a particle that travels from infinity and back out in the small deflection limit in terms of r_0 and the velocity (v) at r_0.

Geodesics around stars (4)

- Furthermore, we can define L in terms of the maximum radius (r_1) by solving for the zero of the bracketed expression
$$\frac{1}{L^2} = \frac{r_1 r_0 \left(r_1 + r_0\right) - 2M\left(r_1^2 + r_0^2 + r_1 r_0\right)}{2m^2 M r_1^2 r_0^2}.$$
- This yields
$$\left(\frac{d\phi}{dr}\right)^2 = \frac{1}{r^2}\frac{r_1 r_0}{\left(r_1 - r\right)\left(r - r_0\right)}\left[1 - 2M\left(\frac{1}{r_0} + \frac{1}{r_1} + \frac{1}{r}\right)\right]^{-1}.$$
- Homework: Use the equation above to find the perihelion advance of Mercury. The advance of perihelion over a given orbit is given by
$$\Delta\phi = 2\int_{r_0}^{r_1} \frac{d\phi}{dr} dr - 2\pi.$$

Beyond the Classical Tests

- Classical: gravitational redshift of light, gravitational lensing, perihelion advance.
- Post-classical:
  - Shapiro delay: light takes longer to travel through the gravitational well of an object
  - Gravitational waves: changes in the field travel at the speed of light
  - Frame dragging: spinning massive objects change the kinematics of spinning objects nearby
  - Orbital decay: orbital energy decreases due to the emission of gravitaional waves

Testing the Assumptions

- General relativity seems to appear out of whole cloth as "the relativistic theory of gravity," so how to quantify the deviations if any. What to they mean?
- There are alternatives:
  - Gauge gravity: Einstein-Cartan-Kibble theory, GTG (except for torision, these are operationally equivalent to GR)
  - Scalar-Tensor gravity: Brans-Dicke, $f(R)$ (these have fifth forces and break the weak equivalence principle)
  - Bimetric theories: one for gravity and one for kinematics
  - Quantum gravity: string theory

Quantifying the Differences (1)

- Strong equivalence principle: gravitational energy acts the same as other types in a gravitational field.
- Weak equivalence principle: all non-gravitational energy acts the same in a gravitational field.
- Is space curved? $\gamma - 1$
- Non-linearity: $\beta - 1$
- Preferred Frames: $\alpha_1, \alpha_2, \alpha_3$
- Failure of energy, momentum and angular momentum conservation: $\zeta_1, \zeta_2, \zeta_3, \zeta_4, \alpha_3$
- Radial vs. Transverse Stress: ξ

Quantifying the Differences (2)

$$\begin{aligned}
g_{00} &= -1 + 2U - 2\beta U^2 - 2\xi \Phi_W + (2\gamma + 2 + \alpha_3 + \zeta_1 - 2\xi)\Phi_1 \\
&\quad + 2(3\gamma - 2\beta + 1 + \zeta_2 + \xi)\Phi_2 + 2(1 + \zeta_3)\Phi_3 \\
&\quad + 2(3\gamma + 3\zeta_4 - 2\xi)\Phi_4 - (\zeta_1 - 2\xi)A \\
&\quad - (\alpha_1 - \alpha_2 - \alpha_3)w^2 U - \alpha_2 w^i w^j U_{ij} \\
&\quad + (2\alpha_3 - \alpha_1)w^i V_i + O(\epsilon^3) \\
g_{0i} &= -\tfrac{1}{2}(4\gamma + 3 + \alpha_1 - \alpha_2 + \zeta_1 - 2\xi)V_i \\
&\quad - \tfrac{1}{2}(1 + \alpha_2 - \zeta_1 + 2\xi)W_i - \tfrac{1}{2}(\alpha_1 - 2\alpha_2)w^i U \\
&\quad - \alpha_2 w^j U_{ij} + O(\epsilon^{\frac{5}{2}}) \\
g_{ij} &= (1 + 2\gamma U)\delta_{ij} + O(\epsilon^2)
\end{aligned}$$