A unifying description of Dark Energy & Modified Gravity

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Astroparticules
et Cosmologie

Outline

1. Introduction
2. Main formalism
3. Linear perturbations
4. Horndeski's theories and beyond
5. Link with observations

Based on

J. Gleyzes, D.L., F. Piazza & F. Vernizzi: 1304.4840, 1404.6495,1408.1952

J. Gleyzes, D.L. & F. Vernizzi: 1411.3712 (review + a few extensions)

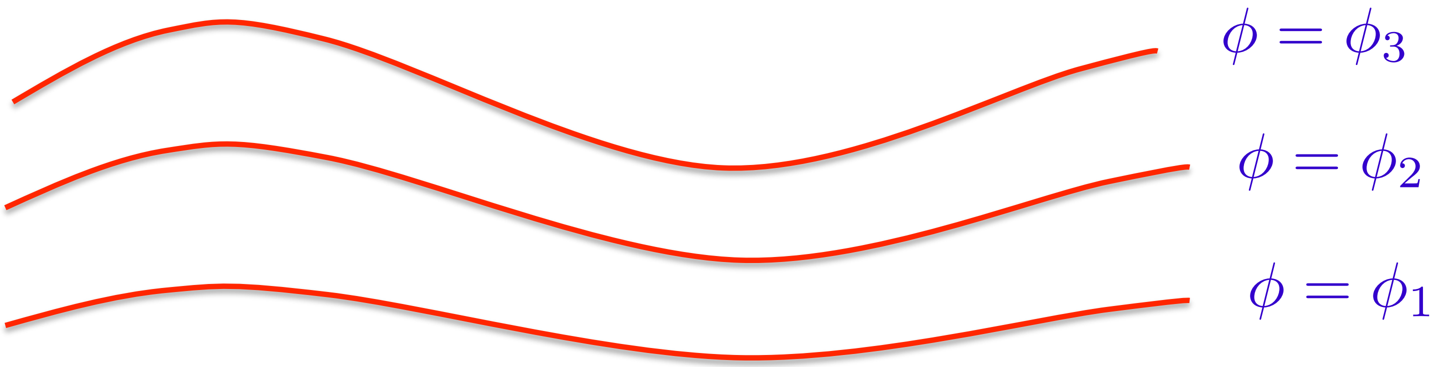
Introduction & motivations

- **Plethora of dark energy models**:
  - Dynamical dark energy: quintessence, K-essence
  - Modified gravity
- Large amount of data from future large scale cosmological surveys (LSST, Euclid, …)
- Goal: **effective description** as a bridge between models and observations.
- Assumptions:
  - **Single scalar field** models
  - All matter fields minimally coupled to the same metric $g_{\mu\nu}$

ADM approach

- The scalar field defines a **preferred slicing**

  Constant time hypersurfaces = uniform field hypersurfaces



- **ADM decomposition** based on this preferred slicing

Uniform scalar field slicing

- **Basic ingredients**
  - Unit vector normal to the hypersurfaces

$$n_\mu = -\frac{1}{\sqrt{-X}}\nabla_\mu\phi\,, \qquad X \equiv g^{\rho\sigma}\,\nabla_\rho\phi\,\nabla_\sigma\phi$$

  - Projection on the hypersurfaces: $h_{\mu\nu} = g_{\mu\nu} + n_\mu\, n_\nu$

  - Intrinsic curvature tensor ${}^{(3)}R_{\mu\nu}$

  - Extrinsic curvature tensor $K_{\mu\nu} = h_{\mu\sigma}\nabla^\sigma n_\nu \qquad K = \nabla_\mu n^\mu$

ADM formulation

- **ADM decomposition of spacetime**

$$ds^2 = -N^2 dt^2 + h_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right)$$

Inverse metric
$$g^{\mu\nu} = \begin{pmatrix} -1/N^2 & N^j/N^2 \\ N^i/N^2 & h^{ij} - N^i N^j/N^2 \end{pmatrix}$$

Hence
$$X \equiv g^{\mu\nu}\,\nabla_\mu\phi\,\nabla_\nu\phi = g^{00}\,\dot{\phi}^2 = -\frac{\dot{\phi}^2(t)}{N^2}$$

- **Lagrangians of the form**

$$S_g = \int d^4x N\sqrt{h}\,L(N, K_{ij}, R_{ij}, h_{ij}, D_i; t)$$

Example: GR + quintessence

- Consider a quintessence model

$$S = \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}\ R - \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi)\right]$$

- For the Einstein-Hilbert term, one can use

$${}^{(4)}R = K_{\mu\nu}K^{\mu\nu} - K^2 + R + 2\nabla_\mu(Kn^\mu - n^\rho\nabla_\rho n^\mu)$$

- In the uniform ϕ slicing, this leads to the Lagrangian

$$L = \frac{M_{\rm Pl}^2}{2}\left[K_{ij}K^{ij} - K^2 + R\right] + \frac{\dot{\phi}^2(t)}{2N^2} - V(\phi(t))$$

Homogeneous evolution

- FLRW metric: $$ds^2 = -N^2(t)dt^2 + a^2(t)\delta_{ij}dx^i dx^j$$
- Extrinsic curvature: $$K^i_j = \frac{\dot{a}}{\bar{N}a}\delta^i_j \equiv H\,\delta^i_j$$
- Homogeneous Lagrangian
$$\bar{L}(a, \dot{a}, \bar{N}) \equiv L\left[K^i_j = \frac{\dot{a}}{\bar{N}a}\,\delta^i_j, R^i_j = 0, N = \bar{N}(t)\right]$$
- One can include **matter** by adding the Lagrangian for matter, minimally coupled to the metric.

Friedmann equations

- Variation of the action $\bar{S}_g = \int dt\, d^3x\, \bar{N} a^3\, \bar{L}(a, \dot{a}, \bar{N})$

$$\delta \bar{S}_g = \int dt d^3x \left\{ a^3 \left(\bar{L} + \bar{N} L_N - 3H\mathcal{F}\right) \delta \bar{N} + 3a^2 \bar{N} \left(\bar{L} - 3H\mathcal{F} - \frac{\dot{\mathcal{F}}}{\bar{N}} \right) \delta a \right\}$$

with $\left(\frac{\partial L}{\partial K_{ij}}\right)_{\mathrm{bgd}} \equiv \mathcal{F}\, \bar{h}^{ij}$

$$\delta \bar{S}_{\mathrm{m}} = \int dt d^3x \bar{N} a^3 \left(-\rho_{\mathrm{m}} \frac{\delta \bar{N}}{\bar{N}} + 3p_{\mathrm{m}} \frac{\delta a}{a} \right) \quad \left[\delta S_{\mathrm{m}} = \frac{1}{2} \int d^4x \sqrt{-g}\, T^{\mu\nu}\, \delta g_{\mu\nu} \right]$$

- **Friedmann equations**

$$\bar{L} + \bar{N} L_N - 3H\mathcal{F} = \rho_{\mathrm{m}} \qquad \bar{L} - 3H\mathcal{F} - \frac{\dot{\mathcal{F}}}{\bar{N}} = -p_{\mathrm{m}}$$

Simple example: K-essence

Armendariz-Picon et al. 00

- K-essence + GR

$$L = \frac{M_{\rm Pl}^2}{2}\left[K_{ij}K^{ij} - K^2 + R\right] + P(X,\phi) \qquad X = -\frac{\dot\phi^2}{N^2}$$

- Background Lagrangian and its derivatives:

$$\bar L = P - 3M_{\rm Pl}^2 H^2 \qquad L_N = -2XP_X$$

$$\frac{\partial L}{\partial K_{ij}} = M_{\rm Pl}^2\left(K^{ij} - Kh^{ij}\right) \quad \Longrightarrow \quad \mathcal{F} = -2M_{\rm Pl}^2 H$$

- Friedmann equations

$$\bar L + \bar N L_N - 3H\mathcal{F} = 3M_P^2 H^2 + P - 2XP_X = \rho_m$$

$$\bar L - 3H\mathcal{F} - \dot{\mathcal{F}} = M_P^2\left(3H^2 + 2\dot H\right) + P = -p_m$$

Linear perturbations

- Perturbations

$$\delta N \equiv N - \bar{N}\,, \qquad \delta K_i^j \equiv K_i^j - H\delta_i^j$$

- Expand the Lagrangian up to quadratic order:

$$L(N, K_j^i, \mathcal{R}_j^i, \dots) = \bar{L} + L_N \delta N + \frac{\partial L}{\partial K_j^i}\delta K_j^i + \frac{\partial L}{\partial R_j^i}\delta R_j^i + L^{(2)} + \dots$$

with

$$L^{(2)} = \frac{1}{2}L_{NN}\delta N^2 + \frac{1}{2}\frac{\partial^2 L}{\partial K_j^i \partial K_l^k}\delta K_j^i \delta K_l^k + \frac{1}{2}\frac{\partial^2 L}{\partial R_j^i \partial R_l^k}\delta R_j^i \delta R_l^k$$

$$+ \frac{\partial^2 L}{\partial K_j^i\, \partial R_l^k}\delta K_j^i \delta R_l^k + \frac{\partial^2 L}{\partial N \partial K_j^i}\delta N \delta K_j^i + \frac{\partial^2 L}{\partial N \partial R_j^i}\delta N \delta R_j^i + \dots$$

Linear perturbations

- The coefficients of the quadratic Lagrangian are evaluated on the homogeneous background, e.g.

$$\frac{\partial^2 L}{\partial K^j_i \, \partial K^l_k} \equiv \hat{\mathcal{A}}_K \, \delta^i_j \, \delta^k_l + \mathcal{A}_K \left(\delta^i_l \, \delta^k_j + \delta^{ik} \delta_{jl}\right)$$

$$\frac{\partial^2 L}{\partial R^j_i \, \partial R^l_k} \rightarrow (\hat{\mathcal{A}}_R, \mathcal{A}_R) \qquad \frac{\partial^2 L}{\partial K^j_i \, \partial R^l_k} \rightarrow (\hat{\mathcal{C}}, \mathcal{C}) \quad \ldots$$

- For simplicity, assume the three conditions

$$\hat{\mathcal{A}}_K + 2\mathcal{A}_K = 0 \, , \quad \hat{\mathcal{C}} + \frac{1}{2}\mathcal{C} = 0 \, , \quad 4\hat{\mathcal{A}}_R + 3\mathcal{A}_R = 0$$

to ensure that the EOM are 2nd order, then the quadratic action depends on only **five time-dependent functions**.

Linear perturbations

- Action at quadratic order

$$S^{(2)} = \int dx^3 dt\, a^3 \,\frac{M^2}{2}\Big[\delta K^i_j \delta K^j_i - \delta K^2 + \alpha_K H^2 \delta N^2 + 4\,\alpha_B H\,\delta K\,\delta N + (1+\alpha_T)\delta_2\left(\frac{\sqrt{h}}{a^3}R\right) + (1+\alpha_H)R\,\delta N\Big]$$

where the alpha's [Bellini & Sawicki's notation] are explicitly given in terms of the derivatives of the Lagrangian.

- GR: $M = M_P, \quad \alpha_i = 0$
- Quintessence, K-essence: $\alpha_K \neq 0$
- Brans-Dicke, F(R): $M = M(t)$
- Kinetic braiding: $\alpha_B \neq 0$
- Horndeski ($\alpha_H = 0$) and beyond Horndeski ($\alpha_H \neq 0$)

Linear degrees of freedom

- Scalar & tensor perts: $h_{ij} = a^2(t)\, e^{2\zeta}\left(\delta_{ij} + \gamma_{ij}^{\mathrm{TT}}\right)$

- Quadratic action for the true degrees of freedom:

$$S^{(2)} = \frac{1}{2}\int dx^3 dt\, a^3 \left[\mathcal{L}_{\dot\zeta\dot\zeta}\dot\zeta^2 + \mathcal{L}_{\partial\zeta\partial\zeta}\frac{(\partial_i\zeta)^2}{a^2} + \frac{M^2}{4}\dot\gamma_{ij}^2 - \frac{M^2}{4}(1+\alpha_T)\frac{(\partial_k\gamma_{ij})^2}{a^2}\right]$$

$$\mathcal{L}_{\dot\zeta\dot\zeta} \equiv M^2\frac{\alpha_K + 6\alpha_B^2}{(1+\alpha_B)^2}\,, \quad \mathcal{L}_{\partial\zeta\partial\zeta} \equiv 2M^2\left\{1 + \alpha_T - \frac{1+\alpha_H}{1+\alpha_B}\left(1+\alpha_M - \frac{\dot H}{H^2}\right) - \frac{1}{H}\frac{d}{dt}\left(\frac{1+\alpha_H}{1+\alpha_B}\right)\right\}$$

- Stability
  - No ghost: $\mathcal{L}_{\dot\zeta\dot\zeta} > 0\,, \qquad M^2 > 0$
  - No gradient instability: $c_s^2 \equiv -\dfrac{\mathcal{L}_{\partial\zeta\partial\zeta}}{\mathcal{L}_{\dot\zeta\dot\zeta}} > 0\,, \quad c_T^2 \equiv 1 + \alpha_T > 0$

Horndeski theories

Horndeski 74; Nicolis et al. 08; Deffayet et al. 09 & 11

- Most general action for a scalar field leading to at most second order equations of motion (Horndeski '74)

  Combination of the following four Lagrangians

$$
\begin{aligned}
L_2^H &= G_2(\phi, X) \\
L_3^H &= G_3(\phi, X)\,\Box\phi \\
L_4^H &= G_4(\phi, X)\,{}^{(4)}R - 2G_{4X}(\phi, X)(\Box\phi^2 - \phi^{\mu\nu}\phi_{\mu\nu}) \\
L_5^H &= G_5(\phi, X)\,{}^{(4)}G_{\mu\nu}\phi^{\mu\nu} + \frac{1}{3}G_{5X}(\phi, X)(\Box\phi^3 - 3\,\Box\phi\,\phi_{\mu\nu}\phi^{\mu\nu} + 2\,\phi_{\mu\nu}\phi^{\mu\sigma}\phi^{\nu}{}_{\sigma})
\end{aligned}
$$

with $\phi_{\mu\nu} \equiv \nabla_\nu \nabla_\mu \phi$

- **ADM formulation** ?

Horndeski into ADM form

Goal: translate the Lagrangian that depend on scalar field derivatives into an ADM expression.

$$\phi_\mu = -\sqrt{-X}\, n_\mu$$

$$\phi_{\mu\nu} = -\sqrt{-X}(K_{\mu\nu} - n_\mu a_\nu - n_\nu a_\mu) - \frac{1}{2\sqrt{-X}}\, n_\mu\, n_\nu\, n^\lambda \nabla_\lambda X$$

with $\quad a^\mu \equiv n^\lambda \nabla_\lambda n^\mu$

Gauss-Codazzi relations are also useful.

$$R_{\mu\nu} = ({}^{(4)}R_{\mu\nu})_\parallel + (n^\sigma n^\rho\ {}^{(4)}R_{\mu\sigma\nu\rho})_\parallel - K K_{\mu\nu} + K_{\mu\sigma} K^\sigma{}_\nu$$

$$R = {}^{(4)}R + K^2 - K_{\mu\nu}K^{\mu\nu} - 2\nabla_\mu (K n^\mu - a^\mu)$$

Horndeski into ADM form

Combination of the Lagrangians

GLPV 1304

$$L_2 = A_2 \qquad L_3 = A_3\, K$$

$$L_4 = A_4\,(K^2 - K_{ij}K^{ij}) + B_4\, R$$

$$L_5 = A_5\,(K^3 - 3KK_{ij}K^{ij} + 2K_{ij}K^{ik}K^j{}_k) + B_5\, K^{ij}\,\left[R_{ij} - h_{ij}R/2\right]$$

with

$$A_2 = G_2 - \sqrt{-X}\int \frac{G_{3\phi}}{2\sqrt{-X}}\,dX\,,$$

$$A_3 = -\int G_{3X}\sqrt{-X}\,dX - 2\sqrt{-X}G_{4\phi}\,,$$

$$A_4 = -G_4 + 2XG_{4,X} + \frac{X}{2}G_{5,\phi}\,,$$

$$A_5 = -\frac{1}{3}(-X)^{\frac{3}{2}}G_{5,X}$$

$$B_4 = G_4 + \sqrt{-X}\int \frac{G_{5\phi}}{4\sqrt{-X}}\,dX\,,$$

$$B_5 = -\int G_{5X}\sqrt{-X}\,dX$$

$$A_4 = -B_4 + 2XB_{4X}$$

$$A_5 = -XB_{5X}/3$$

Beyond Horndeski

GLPV 1404 & 1408

- The Lagrangians

$$L_2 = A_2 \qquad\qquad L_3 = A_3\, K$$

$$L_4 = A_4\,(K^2 - K_{ij}K^{ij}) + B_4\, R$$

$$L_5 = A_5\,(K^3 - 3KK_{ij}K^{ij} + 2K_{ij}K^{ik}K^j{}_k) + B_5\, K^{ij}\,\left[R_{ij} - h_{ij}R/2\right]$$

with generic functions A_4, B_4, A_5 and B_5 do not lead to any additional degree of freedom.

No **Ostrogradski** instabilities !

Beyond Horndeski

- In covariant form, we get extra terms

$$L_4^\phi = G_4(\phi, X)\,{}^{(4)}R - 2G_{4X}(\phi, X)(\Box\phi^2 - \phi^{\mu\nu}\phi_{\mu\nu})$$
$$+ F_4(\phi, X)\epsilon^{\mu\nu\rho}{}_{\sigma}\,\epsilon^{\mu'\nu'\rho'\sigma}\phi_\mu\phi_{\mu'}\phi_{\nu\nu'}\phi_{\rho\rho'}\ ,$$
$$L_5^\phi = G_5(\phi, X)\,{}^{(4)}G_{\mu\nu}\phi^{\mu\nu} + \frac{1}{3}G_{5X}(\phi, X)(\Box\phi^3 - 3\,\Box\phi\,\phi_{\mu\nu}\phi^{\mu\nu} + 2\,\phi_{\mu\nu}\phi^{\mu\sigma}\phi^{\nu}{}_{\sigma})$$
$$+ F_5(\phi, X)\epsilon^{\mu\nu\rho\sigma}\epsilon^{\mu'\nu'\rho'\sigma'}\phi_\mu\phi_{\mu'}\phi_{\nu\nu'}\phi_{\rho\rho'}\phi_{\sigma\sigma'}$$

which do not belong to Horndeski class.

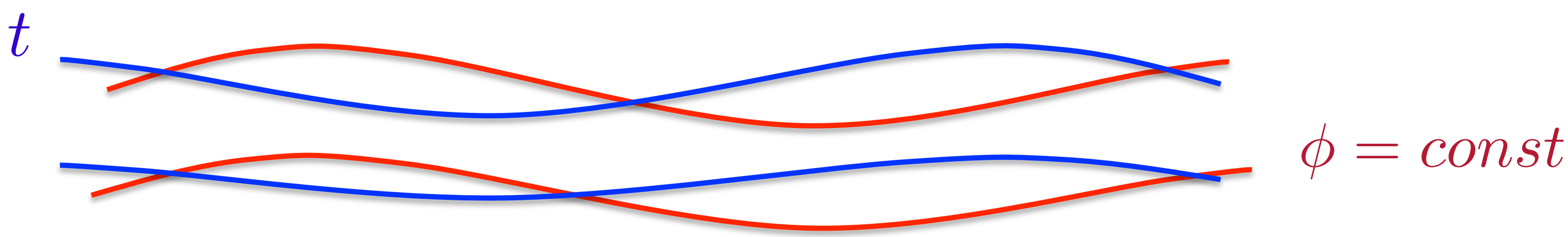
- Lagrangians with $F_4 = 0$ or with $F_5 = 0$ can be connected to Horndeski, via **disformal transformations**

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \Gamma(\phi, X)\,\partial_\mu\phi\,\partial_\nu\phi$$

GLPV 1408

Perturbations in an arbitrary gauge

- Description in an arbitrary slicing ?



- Transformation $\qquad t \to t + \pi(t, \vec{x})$

Perturbations in an arbitrary gauge

- Stueckelberg trick: $t \to t + \pi(t, \vec{x})$

- The new quadratic action can be derived via the substitutions:

$$\begin{aligned}
f &\to f + \dot{f}\pi + \frac{1}{2}\ddot{f}\pi^2 \,, \\
g^{00} &\to g^{00} + 2g^{0\mu}\partial_\mu\pi + g^{\mu\nu}\partial_\mu\pi\partial_\nu\pi \,, \\
\delta K_{ij} &\to \delta K_{ij} - \dot{H}\pi h_{ij} - \partial_i\partial_j\pi \,, \\
\delta K &\to \delta K - 3\dot{H}\pi - \frac{1}{a^2}\partial^2\pi \,, \\
{}^{(3)}R_{ij} &\to {}^{(3)}R_{ij} + H(\partial_i\partial_j\pi + \delta_{ij}\partial^2\pi) \,, \\
{}^{(3)}R &\to {}^{(3)}R + \frac{4}{a^2}H\partial^2\pi \,.
\end{aligned}$$

Note: the 3-dim quantities on the right are defined with respect to the new time hypersurfaces.

Perturbations in the Newtonian gauge

- Perturbed metric

$$ds^2 = -(1 + 2\Phi)dt^2 + a^2(t)\,(1 - 2\Psi)\,\delta_{ij}\,dx^i dx^j$$

- Modified Einstein's equations $\delta G^{\rm mod}_{\mu\nu} = M^{-2}\,\delta T^{\rm matter}_{\mu\nu}$

or $\delta G_{\mu\nu} = M^{-2}\left(\delta T^{\rm matter}_{\mu\nu} + \delta T^{D}_{\mu\nu}\right)$ with

$$\delta p_D = \frac{\gamma_1\gamma_2 + \gamma_3\alpha_B^2\tilde{k}^2}{\gamma_1 + \alpha_B^2\tilde{k}^2}(\delta\rho_D - 3Hq_D) + \frac{\gamma_1\gamma_4 + \gamma_5\alpha_B^2\tilde{k}^2}{\gamma_1 + \alpha_B^2\tilde{k}^2}Hq_D + \gamma_7\delta\rho_{\rm m}$$

$$\sigma_D = \frac{a^2}{2k^2}\left[\frac{\gamma_1\alpha_T + \gamma_8\alpha_B^2\tilde{k}^2}{\gamma_1 + \alpha_B^2\tilde{k}^2}(\delta\rho_D - 3Hq_D) + \frac{\gamma_9\tilde{k}^2}{\gamma_1 + \alpha_B^2\tilde{k}^2}Hq_D + \alpha_T\delta\rho_{\rm m}\right]$$

Testing deviations from GR

- In the **quasi-static approximation** for sub-horizon scales (time derivatives are neglected), one can express

  1. the **effective Newton constant** $$-\frac{k^2}{a^2}\Phi \equiv 4\pi\, G_{\rm eff}(t,k)\, \rho_m \Delta_m$$
  2. the **slip parameter** $$\Psi \equiv \gamma(t,k)\, \Phi$$

  in terms of the coefficients α_i .

- Full system of equations can be implemented in a modified numerical code.

Conclusions

- **Unified treatment** of dark energy and modified gravity models, based on ADM formalism.
  - Easy comparisons between models
  - Identification of degeneracies
  - Observational data can constrain many models simultaneously
  - Explore unchartered territories (e.g. theories beyond Horndeski)

Theoretical models ↔ **Effective description** ↔ **Observations**

- Very general and efficient way to describe linear perturbations in scalar-tensor theories with **only five time-dependent functions**.