Disformal invariance of cosmological perturbations in a generalized class of Horndeski theories

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De Felice and ST, arXiv: 1411.0736
ST, arXiv: 1412.6210

Motivation of going beyond General Relativity

- Origin of inflation

  --it comes from some geometric effect or from a scalar field beyond the standard model of particle physics?

- Origin of dark energy (and dark matter)

  --the present cosmic acceleration may come from a large-distance modification of gravity?

- Construction of renormalizable theory of gravity

  --short-distance modification of gravity

Horndeski theories

$$S = \int d^4x \sqrt{-g}\, L$$

Horndeski (1974)
Deffayet et al (2011)
Charmousis et al (2011)
Kobayashi et al (2011)

$$\begin{aligned} L \;=\;& G_2(\phi, X) + G_3(\phi, X)\Box\phi + G_4(\phi, X)R - 2G_{4,X}(\phi, X)\left[(\Box\phi)^2 - \phi^{;\mu\nu}\phi_{;\mu\nu}\right] \\ & + G_5(\phi, X)G_{\mu\nu}\phi^{;\mu\nu} + \frac{1}{3}G_{5,X}(\phi, X)[(\Box\phi)^3 - 3(\Box\phi)\phi_{;\mu\nu}\phi^{;\mu\nu} + 2\phi_{;\mu\nu}\phi^{;\mu\sigma}\phi^{;\nu}{}_{;\sigma}] \end{aligned}$$

Single scalar field ϕ with $X = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$

R and $G_{\mu\nu}$ are the 4-dimensional Ricci scalar and the Einstein tensors, respectively.

Horndeski theories are of second order, so it is free from the Ostrogradski instability.

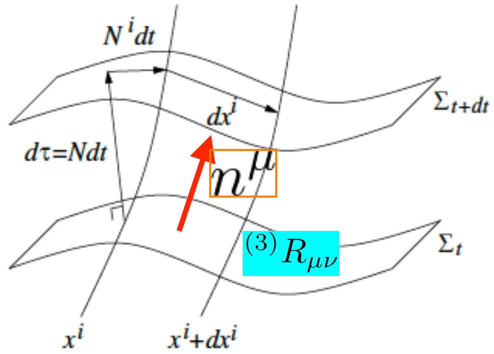
- General Relativity corresponds to $G_4 = M_{\rm pl}^2/2$.
- Horndeski theories accommodate a wide variety of gravitational theories like Brans-Dicke theory, $f(R)$ gravity, and covariant Galileons.

Horndeski Lagrangian in terms of the ADM Language

ADM metric: $ds^2 = -N^2 dt^2 + h_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right)$

N : lapse, N^i : shift n^μ: unit vector orthogonal to Σ_t

- Extrinsic curvature: $K_{\mu\nu} = h^\lambda_\mu n_{\nu;\lambda}$
- Intrinsic curvature: $\mathcal{R}_{\mu\nu} = {}^{(3)}R_{\mu\nu}$



Several scalar quantities can be constructed:

$$K \equiv K^\mu{}_\mu\,, \quad \mathcal{S} \equiv K_{\mu\nu}K^{\mu\nu}\,, \quad \mathcal{R} \equiv \mathcal{R}^\mu{}_\mu\,, \quad \mathcal{Z} \equiv \mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}\,, \quad \mathcal{U} \equiv \mathcal{R}_{\mu\nu}K^{\mu\nu}\,.$$

In the unitary gauge where $\delta\phi = 0$, the Horndeski Lagrangian on the flat FLRW background is equivalent to

$$L = A_2 + A_3 K + A_4(K^2 - \mathcal{S}) + B_4\mathcal{R} + A_5\mathcal{K}_3 - B_5(K\mathcal{R}/2 - \mathcal{U})$$

$$\left(\mathcal{K}_3 = 3H(2H^2 - 2KH + K^2 - \mathcal{S}) + O(3)\right)$$

with two particular relations

$$A_4 = 2XB_{4,X} - B_4, \quad A_5 = -XB_{5,X}/3$$

Gleyzes et al (2013)

Gleyzes-Langlois-Piazza-Vernizzi (GLPV) theories

Consider the Lagrangian

$$L = A_2 + A_3 K + A_4(K^2 - \mathcal{S}) + B_4 \mathcal{R} + A_5 \mathcal{K}_3 - B_5(K\mathcal{R}/2 - \mathcal{U})$$

without putting restrictions between $A_i(N,t)$ and $B_i(N,t)$.

(i) The background and perturbation equations are of second order on the flat FLRW background without an extra propagating degree of freedom.

Gleyzes et al, arXiv:1404.6495, 1408.1952; Lin et al, arXiv: 1408.0670.

(ii) The theories with same A_i but with different B_i lead to the same background dynamics, but the evolution of cosmological perturbations is different.

Kase and ST, PRD90, 044073 (2014)

(iii) Even in the presence of a small anisotropy on the flat FLRW background, it seems that there is no extra degree of freedom.

De Felice, Kase, Mukohyama, ST, in preparation

Disformal transformation

The structure of the GLPV action is preserved under the disformal transformation

$$\hat{g}_{\mu\nu} = \Omega^2(\phi) g_{\mu\nu} + \Gamma(\phi, X) \nabla_\mu \phi \nabla_\nu \phi$$

Conformal transformation — Disformal transformation

In unitary gauge the GLPV action in the transformed frame reads

$$S = \int d^4x \sqrt{-\hat{g}}\, \hat{L}$$

$$\begin{aligned} \hat{L} &= \hat{A}_2(\hat{N},t) + \hat{A}_3(\hat{N},t)\hat{K} + \hat{A}_4(\hat{N},t)(\hat{K}^2 - \hat{\mathcal{S}}) + \hat{B}_4(\hat{N},t)\hat{\mathcal{R}} \\ &\quad + \hat{A}_5(\hat{N},t)\hat{\mathcal{K}}_3 + \hat{B}_5(\hat{N},t)\left(\hat{\mathcal{U}} - \hat{K}\hat{\mathcal{R}}/2\right) \end{aligned}$$

with the relations among coefficients

$$\begin{cases} \hat{A}_2 = \dfrac{1}{\Omega^3 \alpha}\left(A_2 - \dfrac{3\omega}{N} A_3 + \dfrac{6\omega^2}{N^2} A_4 - \dfrac{6\omega^3}{N^3} A_5\right), \\ \hat{A}_3 = \dfrac{1}{\Omega^3}\left(A_3 - \dfrac{4\omega}{N} A_4 + \dfrac{6\omega^2}{N^2} A_5\right), \\ \hat{A}_4 = \dfrac{\alpha}{\Omega^3}\left(A_4 - \dfrac{3\omega}{N} A_5\right), \qquad \hat{B}_4 = \dfrac{1}{\Omega\alpha}\left(B_4 + \dfrac{\omega}{2N} B_5\right), \\ \hat{A}_5 = \dfrac{\alpha^2}{\Omega^3} A_5\,, \qquad \hat{B}_5 = \dfrac{1}{\Omega} B_5\,. \end{cases}$$

where

$$\alpha \equiv \hat{N}/N = \sqrt{\Omega^2 + \Gamma X}$$

$$\omega \equiv \dot{\Omega}/\Omega$$

Gleyzes et al,
arXiv: 1408.1952

Disformal invariance of cosmological perturbations

ST (2014),
See also
Minamitsuji (2014)
for the case $\Gamma = \Gamma(\phi)$

Consider the perturbed metric

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$
$$= -(1+2A)dt^2 + 2\psi_{|i} dt dx^i + a^2(t) q_{ij} dx^i dx^j$$

where

$$h_{ij} = a^2(t) q_{ij}\,, \qquad q_{ij} \equiv (1+2\zeta)\delta_{ij} + \gamma_{ij} + 2E_{|ij}\,.$$

Curvature perturbations (ζ) — Gravitational waves (γ_{ij})

In the unitary gauge ($\delta\phi = 0$), the line element after the disformal transformation $\hat{g}_{\mu\nu} = \Omega^2(\phi) g_{\mu\nu} + \Gamma(\phi, X)\nabla_\mu \phi \nabla_\nu \phi$ reads

$$d\hat{s}^2 = -\hat{N}^2 dt^2 + \hat{a}^2(t)\hat{q}_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$

where

$$\hat{N} = N\sqrt{\Omega^2 + \Gamma X}, \qquad \hat{a}(t) = \Omega(t) a(t), \qquad \hat{q}_{ij} = q_{ij}$$

The invariance of q_{ij} means that

$$\hat{\zeta} = \zeta\,, \qquad \hat{\gamma}_{ij} = \gamma_{ij}$$

Gravitational waves in GLPV theories

Expansion of the GLPV action up to second order in tensor perturbations leads to

$$S_2^{(h)} = \int d^4x \mathcal{L}_2^{(h)} \qquad \mathcal{L}_2^{(h)} = a^3 q_{\rm t} \left[\dot{\gamma}_{ij}^2 - \frac{c_{\rm t}^2}{a^2} (\partial \gamma_{ij})^2 \right]$$

$$q_{\rm t} = \frac{L_{,\mathcal{S}}}{4N}, \quad c_{\rm t}^2 = \frac{N^2 \mathcal{E}}{L_{,\mathcal{S}}}$$

where

$$\mathcal{E} = L_{,\mathcal{R}} + \frac{\dot{L}_{,\mathcal{U}}}{2N} + \frac{3}{2} H L_{,\mathcal{U}}$$

$$H = \frac{\dot{a}}{Na}$$

Under the disformal transformation the quantities $L_{,\mathcal{S}}$ and $\mathcal{E}$ transform as

$$\hat{L}_{,\hat{\mathcal{S}}} = \frac{\alpha}{\Omega^3} L_{,\mathcal{S}} \qquad \hat{\mathcal{E}} = \frac{1}{\Omega \alpha} \mathcal{E}$$

where $\alpha = \hat{N}/N$

$$\Rightarrow \quad \hat{q}_{\rm t} = \frac{1}{\Omega^3} q_{\rm t}, \qquad \hat{c}_{\rm t}^2 = \Omega^2 c_{\rm t}^2$$

The second-order Lagrangian in the transformed frame is of the same form as that in the original frame:

$$\hat{\mathcal{L}}_2^{(h)} = \hat{a}^3 \hat{q}_{\rm t} \left[\dot{\gamma}_{ij}^2 - \frac{\hat{c}_{\rm t}^2}{\hat{a}^2} (\partial \gamma_{ij})^2 \right]$$

Inflationary tensor power spectrum in GLPV theories

The tensor equation of motion in the original frame is

$$\frac{d}{dt}(a^3 q_{\rm t}\dot{\gamma}_{ij}) - a q_{\rm t} c_{\rm t}^2 \partial^2 \gamma_{ij} = 0$$

Consider an inflationary background in which the expansion rate $h = NH = \dot{a}/a$ is nearly constant, i.e.,

$$\epsilon_h \equiv -\frac{\dot{h}}{h^2} \ll 1$$

We introduce the following two parameters

$$\epsilon_{\rm t} \equiv \frac{\dot{q}_{\rm t}}{h q_{\rm t}}, \qquad s_{\rm t} \equiv \frac{\dot{c}_{\rm t}}{h c_{\rm t}}$$

Provided that these terms are much smaller than 1, we obtain the inflationary tensor power spectrum up to next-to-leading order in slow-roll:

$$\mathcal{P}_h(k) = \frac{h^2}{4\pi^2 q_{\rm t} c_{\rm t}^3}\left[1 - 2(C+1)\epsilon_h - C\epsilon_{\rm t} - (3C+2)s_{\rm t}\right]\Big|_{c_{\rm t}k=ah}$$

De Felice and ST, arXiv: 1411.0736

where $C = \gamma - 2 + \ln 2 = -0.7296...$

Transformation to the GR form of the tensor spectrum

In the transformed frame the tensor power spectrum is invariant:

$$\hat{\mathcal{P}}_h(k) = \frac{\hat{N}^2 \hat{H}^2}{4\pi^2 \hat{q}_t \hat{c}_t^3} \left[1 - 2(C+1)\hat{\epsilon}_h - C\hat{\epsilon}_t - (3C+2)\hat{s}_t\right]\Big|_{\hat{c}_t k = \hat{a}\hat{h}}$$

Let us consider a frame satisfying the two conditions

$$\hat{q}_t = \frac{M_{\rm pl}^2}{8\hat{N}}, \qquad \hat{c}_t = \hat{N} \quad \Rightarrow \quad \Omega^2 = \frac{8 q_t c_t}{M_{\rm pl}^2}, \qquad \Gamma = \frac{8 q_t c_t}{M_{\rm pl}^2} \frac{c_t^2 - N^2}{N^2 X}$$

In this case we have

$$\hat{\epsilon}_t = -\hat{s}_t, \qquad \hat{\epsilon}_h = \hat{\epsilon}_H - \hat{s}_t \qquad \text{where} \qquad \hat{\epsilon}_H \equiv -\frac{1}{\hat{N}} \frac{\dot{\hat{H}}}{\hat{H}^2}$$

Then the tensor power spectrum reads

$$\hat{\mathcal{P}}_h(k) = \frac{2\hat{H}^2}{\pi^2 M_{\rm pl}^2} \left[1 - 2(C+1)\hat{\epsilon}_H\right]\Big|_{k = \hat{a}\hat{H}}$$

ST, arXiv: 1412.6210

→ Same as the GR tensor spectrum (Stewart and Lyth, 1993)

Einstein frame

The choice $\hat{q}_t = M_{\rm pl}^2/(8\hat{N})$ corresponds to $\hat{L}_{,\hat{\mathcal{S}}} = M_{\rm pl}^2/2$, i.e.,

$$\hat{A}_4 = -M_{\rm pl}^2/2 - 3\hat{H}\hat{A}_5$$

In this case the background equations of motion can be written as the GR form:

$$\begin{aligned} 3M_{\rm pl}^2\hat{H}^2 &= \hat{\rho}, \\ -2M_{\rm pl}^2\frac{1}{\hat{N}}\frac{d\hat{H}}{dt} &= \hat{\rho}+\hat{P}, \end{aligned}$$

where

$$\hat{\rho} \equiv -\hat{A}_2 - 6\hat{H}^3\hat{A}_5 - \hat{N}\left(\hat{A}_{2,\hat{N}} + 3\hat{H}\hat{A}_{3,\hat{N}} - 12\hat{H}^3\hat{A}_{5,\hat{N}}\right),$$

$$\hat{P} \equiv \hat{A}_2 + 6\hat{H}^3\hat{A}_5 - \left(\dot{\hat{A}}_3 - 12\hat{H}\dot{\hat{H}}\hat{A}_5 - 6\hat{H}^2\dot{\hat{A}}_5\right)/\hat{N},$$

The spectral index of the leading-order tensor spectrum $\hat{\mathcal{P}}_h(k) = 2\hat{H}^2/(\pi^2 M_{\rm pl}^2)$ is

$$\hat{n}_t = \frac{2}{\hat{N}}\frac{\dot{\hat{H}}}{\hat{H}^2}$$

The tensor spectrum is red-tilted ($\hat{n}_t < 0$) for $\hat{\rho} + \hat{P} > 0$, i.e.,

$$\dot{\hat{N}}\left(\hat{A}_{2,\hat{N}} + 3\hat{H}\hat{A}_{3,\hat{N}} - 12\hat{H}^3\hat{A}_{5,\hat{N}}\right) + \left(\dot{\hat{A}}_3 - 12\hat{H}\dot{\hat{H}}\hat{A}_5 - 6\hat{H}^2\dot{\hat{A}}_5\right)/\hat{N} < 0$$

Scalar perturbations

The second-order Lagrangian for curvature perturbations ζ takes the same form as the tensor's one with more involved coefficients q_s and c_s:

$$\mathcal{L}_2^{(s)} = a^3 q_s \left[\dot{\zeta}^2 - \frac{c_s^2}{a^2} (\partial \zeta)^2 \right]$$

The disformal invariance of the scalar action also holds with the correspondence

$$\hat{q}_s = \frac{1}{\Omega^3} q_s, \qquad \hat{c}_s^2 = \Omega^2 c_s^2$$

The inflationary scalar power spectrum reads

$$\mathcal{P}_\zeta(k) = \frac{h^2}{8\pi^2 q_s c_s^3} \left[1 - 2(C+1)\epsilon_h - C\epsilon_s - (3C+2)s_s\right]\Big|_{c_s k = ah}$$

$$\epsilon_s = \dot{q}_s/(hq_s)$$

$$s_s = \dot{c}_s/(hc_s)$$

In the Einstein frame the spectrum is the same and it can be expressed as

$$\hat{\mathcal{P}}_\zeta(k) = \frac{q_{tk} c_{tk}^3}{q_s c_s^3} \frac{\hat{H}^2}{\pi^2 M_{pl}^2} \left[1 - 2(C+1)\hat{\epsilon}_H + 2(C+1)\hat{s}_t - C\hat{\epsilon}_s - (3C+2)\hat{s}_s\right]|_{\hat{c}_s k = \hat{a}\hat{h}}$$

where q_{tk} and c_{tk} should be evaluated at $c_t k = ah$.

→ Together with the tensor power spectrum, these general results can be used to put precise constraints on concrete inflationary models.

Summary and outlook

1. The disformal transformation $\hat{g}_{\mu\nu} = \Omega^2(\phi) g_{\mu\nu} + \Gamma(\phi, X) \nabla_\mu \phi \nabla_\nu \phi$ preserves the structure of the GLPV action. In Horndeski theories we require that $\Gamma = \Gamma(\phi)$.

2. The curvature and tensor perturbations are invariant under disformal transformation.

3. We computed the scalar and tensor inflationary power spectra up to next-to-leading order in GLPV theories and showed that it is possible to transform to the Einstein frame with the GR form of the tensor spectrum.

$$\mathcal{P}_h^{\rm lead}(k) = \frac{N^2 H^2}{4\pi^2 q_{\rm t} c_{\rm t}^2} \quad \longrightarrow \quad \hat{\mathcal{P}}_h^{\rm lead}(k) = \frac{2\hat{H}^2}{\pi^2 M_{\rm pl}^2}$$

The disformal transformation that realizes $\hat{q}_{\rm t} = M_{\rm pl}^2/(8\hat{N}), \hat{c}_{\rm t} = \hat{N}$

4. In application to dark energy, it will be also interesting to take into account additional matter (specific kinetic couplings arise in GLPV theories).