Ch. 10.1 Systematics of Renormalization VI. 1

⊗ We've seen how the U.V. divergences in QED amplitudes (reflecting our ignorance of physics / particles at ultra-high energies) can be absorbed into renormalization of the bare parameters of the theory, at least to one-loop order.

⊗ It's not obvious that this procedure will work at higher orders.

⊗ In fact, the proof that QED is "renormalizable" to all orders is quite technical. But we will be able to make this theorem very plausible.

Counting of Ultraviolet Divergences

⊗ The first step is to systematically classify all **amplitudes** that are ultraviolet divergent.

⊗ Notation used to characterize a given **diagram** in QED:

N_e = number of external electron lines

N_γ = external photon lines

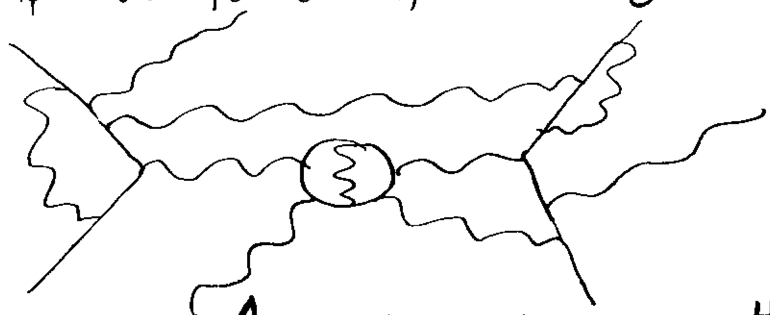
P_e = number of electron propagators

P_γ = photon propagators

V = number of vertices

L = number of closed loops

⊗ The amplitude for a typical diagram:



will go like $A \sim \displaystyle\int^{\Lambda} \frac{d^4k_1\, d^4k_2 \cdots d^4k_L}{\cdots \underbrace{(\not{k}_i + m)^1}_{} \cdots \underbrace{(k_j^2)}_{} \cdots}$

⊗ each **loop** integration "**diverges**" as (some loop momentum)^4

⊗ each e^- propagator "**converges**" as (some loop momentum)^1

⊗ each γ propagator "**converges**" as (some loop momentum)^2

$\boxed{\text{Naively}}$ we expect the diagram to converge if there are more powers of momentum in denominator than numerator

∴ Define the $\boxed{\text{Superficial Degree of Divergence } D}$

$$D = 4L - 1P_e - 2P_\gamma$$

⊗ That is, imagine we impose a cutoff Λ on all loop integrals. On dimensional grounds, we might **naively** expect the amplitude to go like

$$A \sim \frac{1}{D}\Lambda^D \quad \therefore \quad \begin{array}{l} D>0 \text{ naively } \textbf{divergent} \\ D<0 \text{ naively } \textbf{convergent} \end{array}$$

$$\sim \log \Lambda \quad \text{for } D = 0$$

⊛ cf $\int^{\Lambda} \frac{dk}{k}$, which has dimensions Λ^0, but actually $\sim \log \Lambda$

⊛ These naive expectations can be **wrong** for one of 3 reasons

⊛ so no integration ⇒ no divergence! ⊛ this special case pinpoints a flaw in our formula.

(i) **Trivial** diagram with no loops and no propagators has:

$$D = 4 \times \underbrace{0}_{\text{no loops}} - 1 \times \underbrace{0}_{\text{no}} - 2 \times \underbrace{0}_{\text{propagators}} = 0$$

e.g.  has $D=0$, but is <u>actually</u> <u>finite</u>

(<u>all</u> diagrams with $L=0$ are "trivially" convergent because there are no integrations to be done)

∴ Can simply forget about these trivial exceptions to superficial degree of divergence counting rule.

(ii) When a **symmetry** is present, some terms in the integral can cancel, causing the divergence to be "softened" or even eliminated!

e.g.  has $D = 4-1-2\times 1 = 1$, but is <u>actually</u> $\sim \log \Lambda$

$$\left(\text{recall } \Sigma_2 \sim \int d^4k \, \frac{(\not k + m)}{k^2 - m^2} \, \frac{1}{(p-k)^2} \underset{\text{actually}}{\sim} \int d^4k_E \, \frac{1}{k_E^4} \sim \log \Lambda \right)$$

odd at large $\pm k$

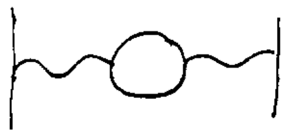
a mathematical "symmetry"!!

e.g.  has $D = 4-2 = 2$, but is <u>actually</u> $\sim \log \Lambda$

(thanks to the Ward Identity; and see below).

∴ Still not to worry: we can still easily identify the <u>actual</u> degree of divergence in such cases, by simply identifying the symmetry from the outset

(iii) When a diagram contains a **divergent subdiagram**, the actual divergence may be **worse** than indicated by D

e.g  has $D = 4\times\underbrace{1}_{\text{1 loop}} - \underbrace{2}_{2\,\bar{e}\text{ props}} - 2\times\underbrace{2}_{2\,\gamma\text{ props}} = -2$

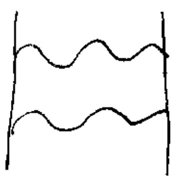
(*) Seems like fatal blow to using D to characterize diagrams

but this diagram actually diverges $\sim \log\Lambda$

(*) **But** the reason for the divergence is "obvious": the inner vacuum polarization loop diverges, regardless of whether it's imbedded in a more complex diagram

(*) This is one of those technical features that we will strongly motivate, but will not prove

$\therefore$ We still don't have to worry about such diagrams, once we've identified the "primitively" divergent amplitudes

e.g  has $D = 4\times\underbrace{1}_{\text{1 loop}} - 1\times\underbrace{2}_{2\,\bar{e}\text{ props}} - 2\times\underbrace{2}_{2\,\gamma\text{ props}} = -2$

and yes this diagram is actually convergent

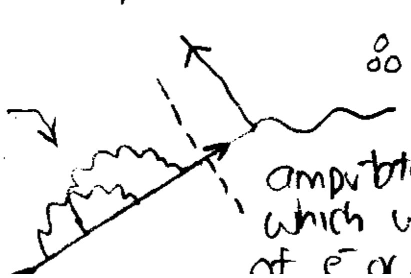
(it has no inner self-energies or vertex corrections)

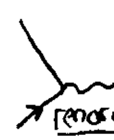
(*) Think of doing each inner integral $\int d^4k_1 \ldots d^4k_L$ until we get to the last one which ~~would diverge if $D \geq 0$. However, if an inner integral diverges by itself, we can~~ go no further until it's regulated

$\boxed{\text{We can also simplify our enumeration problem by}}$

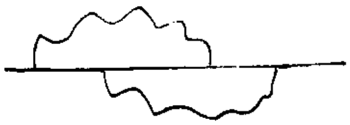
(a) Using the LSZ theorem to recognize that we never actually evaluate diagrams with self-energies on external lines (this serves only to remind us to convert a non-interacting $\bar{e}$ to a self-interacting free $\bar{e}$)

$\therefore$ consider only "amputated" diagrams

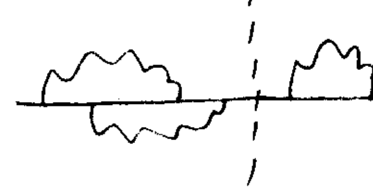
e.g. forget about since all this is self-interaction  amputate at "last" propagator which would "blow-up" because of $\bar{e}$ or γ being on-shell

$\therefore$ diagram actually "reads":  renormalized electron

(b) Consider only "one-particle irreducible" (1PI) diagrams: these are diagrams that <u>cannot</u> be split in two by removing a single line.

e.g  is 1PI, while

split

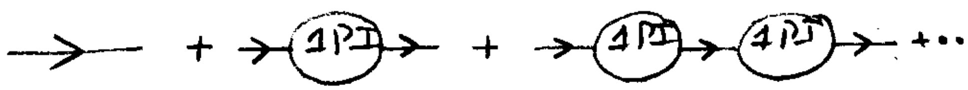
 is <u>**not**</u> 1PI.

⊛ We can forget about non-1PI diagrams ("reducible" diagrams) because the individual pieces simply multiply, so the divergences come from the individual 1PI components.

By the way, we can <u>uniquely</u> divide <u>all</u> Feynman diagrams into 1PI and non-1PI diagrams. This then provides some justification for the geometric series summation that we did to compute the renormalized e^- and γ propagators:

⊛ Define $-i\,\Sigma_{\text{exact}}(p) =$ 

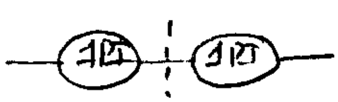
$\equiv$  (as usual, we don't count external electron lines in defining Σ)

∴ $S_{F,\text{exact}}(p) =$ 

= sum of <u>all</u> Feynman diagrams with one electron line in and one out:

$$= \frac{i}{\not p - m_0 - \Sigma'(p)}$$

just as we did for lowest order calculation $-i\Sigma'_2 =$ 

⊛ Note that the 1PI "cut" could only be at a "lone" electron line, e.g , since otherwise we violate e.g. charge conservation in split diagrams

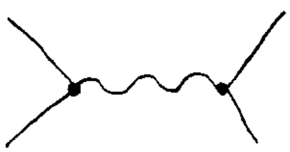
(*) To finally characterize all possible divergent **amplitudes**, we derive $\boxed{\text{another, more convenient, expression for } D}$

(*) First, use : $$L = P_e + P_\gamma - V + 1$$

[Recall that in deriving momentum space Feynman rules from Wick's theorem for S-matrix : we Fourier-transformed each coordinate space propagator, giving one associated loop integration ; But each vertex had a 4-momentum conserving δ^4-function, which eliminates some of the integrations, except for an overall external 4-momentum conserving δ^4-function]

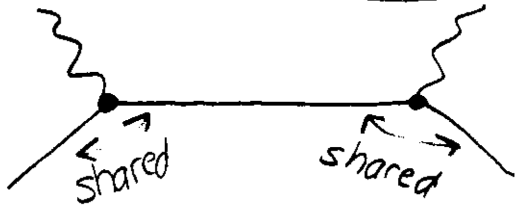
(*) Next, use : $$V = 2P_\gamma + N_\gamma$$

i.e. each photon propagator connects to 2 vertices, while each external (or "real") photon line connects to 1 vertex :

e.g.  and 

(*) Finally, use : $$V = \frac{1}{2}\left(2P_e + N_e\right)$$

i.e. because of charge conservation (one e^- in = one e^- out) we now have each electron propagator sharing 2 vertices and each external electron sharing 1 vertex (i.e. either with another propagator or external line) :

e.g. 

$$\therefore\ D = 4L - P_e - 2P_\gamma$$

$$= 4(P_e + P_\gamma - V + 1) - P_e - 2P_\gamma$$

$$= 3P_e + 2P_\gamma - 4V + 4$$

$$= (3P_e - 3V) + (2P_\gamma - V) + 4$$

$$= \left(3\cancel{P_e} - 3\cancel{P_e} - \tfrac{3}{2}N_e\right) + \left(2\cancel{P_\gamma} - 2\cancel{P_\gamma} - N_\gamma\right) + 4$$

$$\therefore\ \boxed{D = 4 - N_\gamma - \tfrac{3}{2}N_e}$$

$\therefore\ N_e = 0 \times \{N_\gamma = 0,1,2,3,4\}$
$N_e = 2 \times \{N_\gamma = 0, 1\}$
where N_e = even only, to conserve charge. (one e^- in must have one e^- out)

⊗ This is very convenient: the superficial degree of divergence of a QED diagram depends only on the number of external lines, not on the # vertices

⊗ So now we can "back off" from individual diagrams and instead consider the very small number of amplitudes (each amplitude corresponding to an ∞ number of Feynman diagrams, each of which may diverge) that can diverge in a **"primitive"** way

⊗ There are in fact only 7 primitive amplitudes with: $D \geq 0$

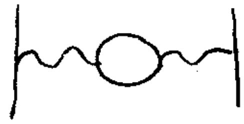
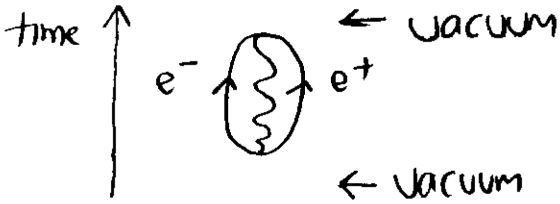
(again, other diagrams or amplitudes may diverge, even those with $D < 0$, but only when they contain one of these 7 as subdiagrams; recall e.g ).

Fig. 10.2(a)  $D = 4$ since no external lines

⊗ Example of a specific diagram that contributes to this amplitude:



= vacuum-to-vacuum transition amplitude

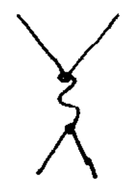
⊗ This is horribly divergent, but it is also irrelevant, because this is a process going on in the vacuum all the time.

e.g. $\mathcal{M}(e^- e^+ \to q\bar{q})$ including vacuum processes =  × [1 +  +  + ⋯]

leading order physical process

= 

⊗ The point is that we should compute physical amplitude, relative to the vacuum-to-vacuum transition amplitude:

$\mathcal{M}_{\text{relative to vacuum}}$ =  (we will hopefully prove this formally in Chapter 9)

∴ We should forget about  : it appears in no physical amplitude

⊗ This points to a flaw in our definition of the S-matrix. We should really have defined:
$S = U(+\infty, -\infty) / \langle vac(T=+\infty) | vac(T=-\infty) \rangle$ [so that $S_{f=vac,\, i=vac} = 1$!]; see P+S Ch. 4.

Fig. 10.2(b)  $D = 4 - 1 = 3$

⊗ The photon "one-point" function (the terminology refers to the "coordinate" of the right-end of the photon line)

⊗ "Charge-conjugation" symmetry implies that this amplitude actually vanishes!

⊗ Here is a <u>sketch</u> of the origin of this symmetry.

⊗ Consider exact QED field equations for A_{op}

$$\partial_\mu F_{op}^{\mu\nu}(x) = e\, \overline{\psi}_{op}(x)\, \gamma^\mu\, \psi_{op}(x)$$

⊛ Ex: Take $\nu=0$ to recover charge density on RHS:

$$\partial_\mu F_{op}^{\mu,\nu=0}(x) = e \sum_{\vec{k},r} \left[c_r^\dagger(\vec{k}) c_r(\vec{k}) - d_r^\dagger(\vec{k}) d_r(\vec{k}) \right]$$

⊗ Recall that this is how we identified the $c^\dagger$ and $d^\dagger$ operators as electrons and positrons, in QFT I.

⊗ However, it is immaterial as to what we call the "fundamental" particle of charge e: electron $(e<0)$ or positron $(e>0)$

⊗ More precisely, there is a symmetry in this theory, defined by the "charge-conjugation" operator C:

$$C\, c_r\, C^{-1} \equiv \tilde{c}_r = d_r$$

$$C\, d_r\, C^{-1} \equiv \tilde{d}_r = c_r$$

<u>and</u> $$C\, a_r\, C^{-1} \equiv \tilde{a}_r = -\, a_r$$

We say that the neutral photon has an "intrinsic" "charge-parity" of (-1)

⊗ photon field operator

where the minus sign last line absorbs the minus sign that occurs on the RHS of Maxwell equations when we flip $c \leftrightarrow d$.

⊗ So, if we start with:

$$\partial_\mu F^{\mu\nu}_{op}(x) = e\,\overline{\Psi}(x)\,\gamma^\mu\,\Psi(y)$$

then after applying $\overrightarrow{C}$ $\overleftarrow{C^{-1}}$, we end up with:

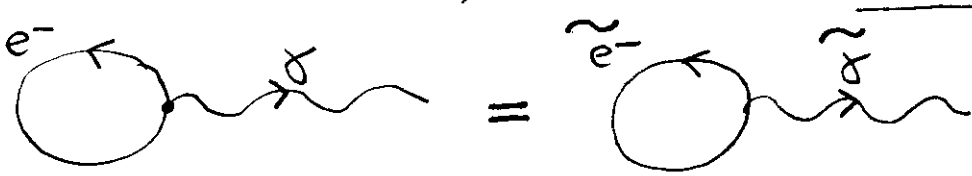
$$\partial_\mu \tilde{F}^{\mu\nu}_{op}(y) = e\,\overline{\tilde{\Psi}}(x)\,\gamma^\mu\,\tilde{\Psi}(y)$$

⊗ That is, we map the original Quantum Field Theory into another QFT of identical structure

[Well, we should also check what happens to the field equation for the Dirac operator; see P+S pp. 70-71, and even better: Bjorken + Drell, RQM, Chapter 5].

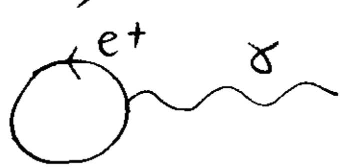
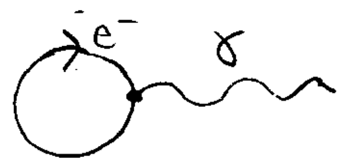
(the amplitudes in original and ~ fields are identical since the field eqns are identical)

⊗ To see the consequences consider the specific diagram:



"original QED" "charge conjugated QED"

⊗ However, loosely speaking: $\tilde{e}^- = e^+$; $\tilde{\gamma} = -\gamma$

∴ = −  = − 

⊗ Finally, the Feynman rule for the closed loop has a Trace, which doesn't care about the direction of the "circulation"
(alternatively, the e^+ and e^- freefield propagators are identical)

∴  = 0 by charge-conjugation symmetry.

(Also, at least some of these "tadpole" diagrams vanish for a normal ordered Hamiltonian: see QFT I, Ch. 7, pg. 149.)

⊛ In fact, the charge-conjugation symmetry generalizes to:

$\boxed{\text{Furry's Theorem}}$

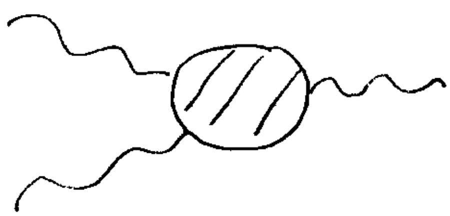
"All n-point photon amplitudes, with n = odd, vanish"

⊛ The argument goes through as above, with n photons acquiring a factor of $(-1)^n$ under charge-conjugation

(by the way, individual Feynman diagrams do not vanish, only the total amplitude)

⊛ You will prove Furry's theorem explicitly in Problem 10.1, using a neat calculational trick.

ஃ $\boxed{\text{Fig. 10.2 (d)}}$



has $D = 4 - 3 = 1$

but this amplitude again vanishes by charge-conjugation symmetry.

(so 1 superficially divergent amplitude eliminated by $\mathcal{M}$ relative to vacuum, 2 amplitudes = 0 by C ∴ 4 left)

$\boxed{\text{Fig. 10.2 (f)}}$

$D = 4 - 2 \times \frac{3}{2} = 1$

⊛ We already saw that the one-loop electron self-energy had 2 ultraviolet divergences, one in the mass renormalization, and one in the residue of the electron propagator after summing the geometric series.

* We now show that these are the only 2 divergences to <u>all</u> <u>orders</u> in perturbation theory.

* The amplitude is a function of the external momentum, so let's do a Taylor expansion about $\not{p} = m$ = renormalized mass

$$= A_0 + A_1(\not{p}-m) + A_2(\not{p}-m)^2 + \cdots$$

where each coefficient is a pure number:

$$A_n = \frac{1}{n!}\frac{d^n}{d\not{p}^n}\Big($$  $$\Big)\Big|_{\not{p}=m}$$

* Now A_0 has the same dimensions as , so it has the <u>same</u> superficial degree of divergence, $D_{A_0}=1$.

* Since A_1 has one unit <u>less</u> of mass dimension, we expect its superficial degree of divergence to be $D_{A_1}=0$.

* Similarly, all $A_{n\geq 2}$ should be superficially convergent

* To see how this would arise mathematically recognize that the external momentum $\not{p}$ will appear in only (unrationalized) propagator denominators inside the loop integrals, giving expressions like:

$$\frac{d}{d\not{p}}\left(\frac{1}{\not{k}+\not{p}-m}\right)^{1} = -\left(\frac{1}{\not{k}+\not{p}-m}\right)^{2}$$

$\uparrow$ some linear combination of loop momenta

∴ Each derivative with respect to an external momentum lowers the superficial degree of divergence by 1.

⊗ Two technical points:

(a) Derivatives can also introduce <u>infrared</u> divergences, as we saw with one-loop electron self-energy. These would have to be regulated say with a <u>temporary</u> photon mass, that disappears when we compute physically meaningful I.R. processes.

(b) As usual, power counting arguments break down when we have divergent sub-diagrams. Don't interchange derivatives and integrals until sub-diagrams are regulated!

⊗ OK, so our naive expectations are:

$$A_0 \sim \Lambda^1, \quad A_1 \sim \log \Lambda, \quad \text{and} \quad A_{n\geq 2} = \text{convergent}$$

⊗ <u>However</u>, <u>odd</u> divergences don't actually occur, because of oddness of the integrand under $k \to -k$

$\left[\text{Recall } \int d^4k_E \frac{k^\mu}{(k^2+\Delta)^n} = 0\right]$

∴ We conclude finally that:

(see also the chiral symmetry argument in P+S above Eq. 10.6)

$$A_0 \sim \underset{\sim}{m} \log \Lambda \quad \text{and} \quad A_2 \sim \log \Lambda$$

(only dimensionful quantity left to give correct units to A_0)

finite dimensionless numbers

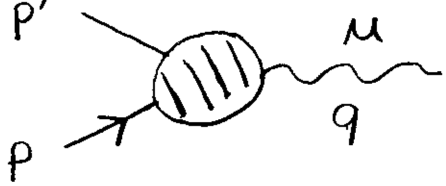
$$= a_0 m \log \Lambda + a_1(\not{p} - m)\log \Lambda + \begin{matrix}\text{finite} \\ \text{terms}\end{matrix}$$

∴ When we (formally) compute the **exact** electron propagator, by summing the geometric series of exact self energies, we ∴ find just the **same** divergences and renormalization, as at one-loop: (up to signs, i's)

⊗ electron mass renormalization: $\delta m = a_0 m \log \Lambda$

⊗ "wavefunction" renormalization: $Z_1^{-1} - 1 = a_1 \log \Lambda$

leading order propagator

$\boxed{\text{Fig. 10.2(g)}}$  $D = 4 - 1 - 2\times\frac{3}{2} = 0$

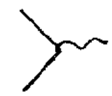
$$\circ\!\!\circ \quad = -ie f_0 \gamma^\mu \log\Lambda \quad + \text{finite terms}$$
(two by momentum conservation?)

⊛ That is, if we expand in powers of the three external momenta, then only the zero derivative term has the divergence (derivatives w.r.t. external momenta again lowering the degree of divergence)

⊛ To justify the γ^μ structure, do the expansion about zero external momenta. Then the only "vector" available for the zero derivative term is γ^μ.

$\circ\!\!\circ$ When we (formally) compute the **exact** vertex function, we find just the **same** divergence as in the one-loop calculation (being a little rough here):

$$Z_1^{-1} - 1 \overset{\text{up to signs and } i\text{'s}}{=} e f_0 \log\Lambda$$

↑ since actually the tree part of , that is , is finite, so $e f_0 \log\Lambda$ actually occurs in ( − ).

⊛ Well, this will be done more systematically in Sect 10.3.

⊛ OK, only 2 more superficially divergent amplitudes left!

$\boxed{\text{Fig. 10.2 (c)}}$  $D = 4 - 2 = 2$

⊛ However, the photon self-energy must satisfy the Ward Identity at <u>all</u> orders in perturbation theory, else when we evaluate the propagators in different gauges, we will get different results for scattering amplitudes

∴ We must demand that:

 $= \left(g^{\mu\nu} q^2 - q^\mu q^\nu\right) \Pi(q^2)$

⊛ So on dimensional grounds we have lowered the degree of divergence by 2 (or, think of having gone to second order in the Taylor expansion about $q^\mu = 0$)

$$\therefore \quad = \left(g^{\mu\nu} q^2 - q^\mu q^\nu\right) c \log \Lambda + \text{finite terms}$$

∴ When we (formally) compute the **exact** photon propagator, by summing the geometric series of exact self energies, we find just the **same** type of divergence as in one-loop:

⊛ $Z_3^{-1} - 1 = c \log \Lambda$

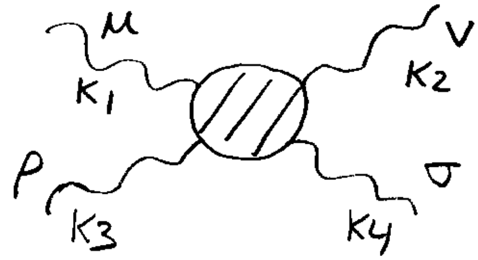
$\uparrow$ from the leading order propagator

⊛ <u>And</u>, thanks to Ward Identity preserving tensor structure of the self-energy, the photon remains <u>massless</u>.

⊛ of course, it is not yet obvious that we <u>can</u> preserve the Ward Identity to all orders (though the algebra on pp. 118-119 makes it look very plausible)

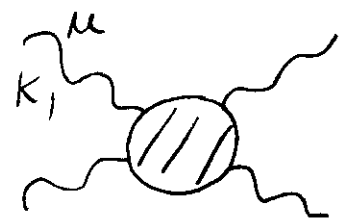
$\boxed{\text{Fig. 10.2 (e)}}$

Last One!



$D = 4 - 4 = 0$

* However, the Ward Identity again lowers the divergence, which makes it **convergent**.

* e.g. k_1^{μ} () $= 0$ and similarly for the other external momenta.

* Again, this condition is required to kill off differences between the photon propagator in different gauges which go like k^{μ} (which is equivalent to the requirement of current conservation for vacuum fluctuations)

* Note that amplitude must be symmetric under interchanges such as $(k_1, \mu) \leftrightarrow (k_2, \nu)$ due to photon statistics [we actually add 6 diagrams with all possible γ permutations] (see Jauch + Rohrlich, pp. 287-292)

* I think the tensorial structure is non-trivial to enumerate, but the following structure highlights the main argument (though I doubt this is unique!):

$$\Big[\big(g^{\mu\nu} k_1^{\sigma} - g^{\mu\sigma} k_1^{\nu}\big) \overset{\nu \leftrightarrow \rho}{+} \big(g^{\mu\rho} k_1^{\sigma} - g^{\mu\sigma} k_1^{\rho}\big) \overset{\rho \leftrightarrow \sigma}{+} \big(g^{\mu\nu} k_1^{\rho} - g^{\mu\rho} k_1^{\nu}\big) + (\nu \leftrightarrow \sigma) \Big]$$

so each ( ) vanishes under contraction with k_1^{μ} and we symmetrized under pairs of the (ν, ρ, σ) polarization labels, keeping (k, μ) fixed

* probably need more interchanges to get full photon symmetrization

$\otimes \big[(k_1, \mu) \leftrightarrow (k_2, \nu)\big] \otimes \big[(k_1, \mu) \leftrightarrow (k_3, \rho)\big] \otimes \big[(k_1, \mu) \leftrightarrow (k_4, \rho)\big]$

* so that amplitude will vanish under contraction with all $k_1^{\mu}, k_2^{\nu}, k_3^{\rho}, k_4^{\sigma}$

* The real point is that to satisfy the Ward Identity for all 4 propagators, we must introduce 4 projectors with the dimensions of mass. ∴ On dimensional (or Taylor series expansion) grounds, the actual divergence is $0 - 4 = -4$

$\boxed{\text{∴ convergent}}$

∴ Divergences in QED to <u>all</u> orders in perturbation theory are identical in structure to one-loop calculations!

⊛ Before we discuss the actual renormalization procedure at all orders:

Consider a $\boxed{\text{More General Characterization of QFTs}}$

⊛ First, let's consider QED in an arbitrary number of spacetime dimensions d (here we are really thinking about <u>physical</u> d = integer)

⊛ Then when we Fourier transform coordinate space propagators, we get d-dimensional loop integrals

$$\therefore \quad D = dL - P_e - 2P_\gamma$$

based on dimensional analysis of the amplitude

(the structure of the momentum space propagators is <u>unchanged</u>, assuming that we don't change the Lagrangian, that is, we keep the same field equations to get the same spin, which is what determines the propagators)

⊛ We <u>still</u> have:

⊛ $L = P_e + P_\gamma - V + 1$ since we have the same # of momentum conserving δ-functions

⊛ $V = 2P_\gamma + N_\gamma = \frac{1}{2}(2P_e + N_e)$ since we draw graphs in the same way.

$$\therefore \quad D = d(P_e + P_\gamma - V + 1) - P_e - 2P_\gamma$$

$$= \underbrace{P_e}_{= V - \frac{1}{2}N_e}(d-1) + \underbrace{P_\gamma}_{= \frac{1}{2}(V - N_\gamma)}(d-2) - dV + d$$

$$\therefore = d + V\left(d - 1 + \tfrac{1}{2}(d-2) - \cancel{d}\right) - N_e\tfrac{1}{2}(d-1) - N_\gamma\tfrac{1}{2}(d-2)$$

$$\boxed{D = d + \left(\frac{d-4}{2}\right) V - \left(\frac{d-2}{2}\right) N_\gamma - \left(\frac{d-1}{2}\right) N_e}$$

- The cancellation of V in this expression is special to $d=4$
- For $d < 4$, the degree of divergence goes <u>down</u> as we go to diagrams with more vertices, so there is only a **finite** number of **diagrams** that diverge!
- For $d > 4$, the degree of divergence gets <u>worse</u> as the number of vertices goes up (meaning e.g. more loops for an amplitude with a given number of external electrons and γ's), so **every amplitude** becomes divergent at some order of perturbation theory Λ!
- These three types of ultra-violet behaviours do characterize more general QFTs (as we will see shortly!):

$\boxed{\text{Super-renormalizable QFT}}$ Only a finite number of Feynman **diagrams** superficially diverge (How computationally happy! However, this doesn't happen in most realistic 4D theories <u>and</u> some "super" theories are actually rather "pathological"!)

"Real" QED $\boxed{\text{Renormalizable QFT}}$ Only a finite number of **amplitudes** are superficially divergent, though divergences do occur at <u>all orders</u> in perturbation theory (Well ok, computationally hard, but at least we need only "renormalize" a finite number of fundamental parameters, so we can make "lots" of predictions)

$\boxed{\text{Non-renormalizable QFT}}$ **All** amplitudes diverge at a sufficiently high order in perturbation theory. So we must renormalize an <u>infinite</u> number of fundamental parameters. Can we still make predictions? YES!!

Yet one more (very deep!) characterization

⊛ Consider Action $S = \int dt\,(T - V)$ say in 1D field theory

⊛ In units of $\hbar = 1$, energy and time have same units

∴ $S_{QFT} = \int d^d x\, \mathcal{L}$ = dimension-less (actually, fixed dimensions of $\hbar$ = angular momentum.)

$\Rightarrow$ Dimensions $(\mathcal{L}) = (\text{mass})^d$

⊛ Now we will preserve the "spin" structure of the QED action [i.e. the field equations are still based on $(P_{op}^2 - \mu^2)$ Boson $= 0$ (so still spin zero) and $(\not{p} - m)$ Fermion $= 0$ (so still spin 1/2)]

∴ $\mathcal{L}^d_{QED} \overset{\text{still}}{=} -\frac{1}{4}(F_{\mu\nu})^2 - \bar{\Psi}(i\not{\partial} - m - e\not{A})\psi$

∴ $\text{Dim}(m\bar{\Psi}\psi) = (\text{mass})^d \Rightarrow \boxed{\text{Dim}(\psi) = (\text{mass})^{\frac{d-1}{2}}}$

$\text{Dim}((\underbrace{\partial_\mu}_{\text{dim (mass)}^1} A_\nu)^2) = (\text{mass})^d \Rightarrow \boxed{\text{Dim}(A) = (\text{mass})^{\frac{d-2}{2}}}$

$\text{Dim}(e\,\underbrace{\bar{\Psi}\not{A}\psi}_{(\text{mass})^{d-1}}) = (\text{mass})^{d} \Rightarrow \boxed{\text{Dim}(e) = (\text{mass})^{\frac{4-d}{2}}}$ $\quad (\text{mass})^{\frac{d}{2}-1}$

⊛ Most interesting: The mass dimension of the QED^d coupling constant seems to be related to the criteria for renormalizability of the theory.

⊛ This is not an accident! To see how the mass dimension of a more general coupling is related to the renormalizability of a general QFT, consider a hypothetical interaction Lagrangian:

$$\mathcal{L}_I = g\left[\bar{\Psi}(\text{matrices})\Psi\right]^{\frac{n_e}{2}}\left[A_{(\text{indices})}\right]^{n_\gamma}$$

⊛ In first order perturbation theory, this causes transitions according to the S-matrix elements:

$$S^{(1)}_{fi} = -\frac{ig}{\hbar}\int d^dx \, \langle f \,|\, T(\cancel{:\mathcal{L}_I:}) \,|\, i\rangle \quad \text{(i.e. no propogators)}$$

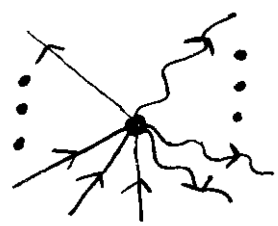
$-\mathcal{L}_I:$

⊛ At first-order, the T-ordering is irrelevant ∴ no "internal contractions"

∴ First-order Amplitude has n_e electrons lines and n_γ photon lines, since all operators must contract with either $\langle f|$ or $|i\rangle$

$$\therefore \mathcal{M}^{(1)}_{(n_e \leftrightarrow n_\gamma)} =$$

n_e electron lines



n_γ photon line

$$= -ig \times (\text{Number of contractions})$$

⊛ Hence the mass dimension of the amplitude is just the dimension of g, which we can work out because $\mathrm{Dim}(\mathcal{L}) = (\text{mass})^d$, and since we know $\mathrm{Dim}(\Psi)$, $\mathrm{Dim}(A)$:

⊛ Why are we fixated on the units of $\mathcal{M}$, and not e.g. $S_{fi} = \mathcal{M} \times \delta^4 \times \sqrt{\frac{1}{2\omega}}$ etc.??
⊙ Because $\mathcal{M}$ (and g) are Lorentz Invariant!!

$$\therefore \mathrm{Dim}(\mathcal{M}) = \mathrm{Dim}(g)$$

$$= (\text{mass})^d \Big/ (\text{mass})^{n_e\left(\frac{d-1}{2}\right)}(\text{mass})^{n_\gamma\left(\frac{d-2}{2}\right)}$$

⊛ Now suppose that we start to compute loop diagrams that contribute to $\mathcal{M}(n_e \leftrightarrow n_\gamma)$.

⊗ Whatever the Feynman rules, all diagrams must necessarily has the same mass dimension, $Dim(g)$.

⊗ If the diagram has V vertices generated by the interaction $\mathcal{L}_I$, and has a superficial degree of divergence D, then by dimensional analysis:

$$Dim\left(g^V \Lambda^D\right) = Dim(g)$$

⊗ This has been derived only for amplitudes that start at first-order.

⊗ The non-renormalizability of interactions with a coupling of negative mass dimension is just a consequence of dimensional analysis. As the number of loops increases ($\therefore V\uparrow$ for fixed N_e, N_γ) higher degrees-of-divergence are **inevitably** introduced $\therefore$ All amplitudes becomes divergent at some order of perturbation theory.

[Note that the divergence could in principle be "softened" if Λ occured in multiplication with some particle mass or external momentum e.g $g^V m^a \Lambda^D$. However, all energies and masses come in through propagator denominator. So, if we regulate with a Pauli-Villars heavy propagator, then Λ is "always" added to m or q^2 etc. $\frac{\text{not multiplied}}{\text{by } \Lambda}$ and hence Λ appears by itself as $\Lambda \to \infty$]

to complete the check of renormalizability (as we have done for QED)

⊗ If $Dim(g) = 0$, then we must also analyze loop diagrams $\wedge$ [i.e. amplitudes like  with Dimension $(mass)^2$]

∴ We conclude that the only interactions that are allowed in (super)-renormalizable QED$_d$ must satisfy:

$$\mathrm{Dim}(g) = (\mathrm{mass})^{d - \frac{n_e}{2}(d-1) - \frac{n_\gamma}{2}(d-2)} = (\mathrm{mass})^{+p}$$

(i.e. positive mass dimension, not necessarily an integer)

[⊗ Note again: if we "pass" this "minimal" criterion, we must still examine amplitudes that start at loop-level 1]

⊗ We recover our previous conclusion for "actual" QED$_d$:

$$\mathcal{L}_I = e\bar{\psi}\gamma^\mu\psi A_\mu \;:\; n_e = 2,\; n_\gamma = 1$$

$$\therefore\; \cancel{d} - (\cancel{d}-1) - \tfrac{1}{2}(d-2) = p \geq 0$$

∴ $d \leq 4$ for "pure" QED to be (super-) renormalizable!

⊗ Another Ex: Take general condition in $d=4$: ∴ Require $4 - \frac{3}{2}n_e - n_\gamma \geq 0$.

⇒ In $d=4$ renormalizable QED allows only $n_e = 2$, $n_\gamma = 1$ i.e. $\mathcal{L}_{\text{pure}} = e\bar{\psi}A\!\!\!/\psi$!!

(*) OK, we haven't surveyed every field theory for renormalizability when loop corrections are included, but these conclusions are very general:

Super-Renormalizable	Coupling constant has positive mass dimension
Renormalizable	Coupling constant is dimensionless
Non-Renormalizable	Coupling constant has negative mass dimension

⊗ At the start of the next section, we will back this up with a simpler field theory where we can easily check the loop diagrams as well, from this dimensional analysis point of view.

A more complete example: Scalar Field Theory

⊛ Let's consider the simplest field theory, with a single scalar field (the Klein-Gordon field) that interacts with itself:

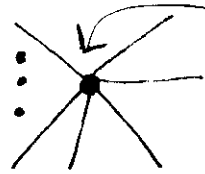
$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m_0^2 \phi^2 - \frac{\lambda_0}{n!} \phi^n$$

⊛ Let's use dimensional analysis to derive all conditions for renormalizability: (including loops) (did not do this part in class)

[We are soon going to specialize to $n = 4$ ("phi-to-the-fourth" theory), which you analyzed in first-order perturbation theory last semester]

⊛ $$S_{fi}^{(1)} = -i \frac{\lambda_0}{n!} \int d^4x \, \langle f | \phi^n | i \rangle$$

so we must have exactly n particles total in i plus f to use up all the contractions between ϕ^n and $\langle f|, |i\rangle$

$\therefore \; \mathcal{M}^{(1)} =$ n external lines  vertex rule $= -i\lambda_0$

⊛ At loop level: propagator attaches to 2 vertices 

$\therefore \quad L = P - V + 1$ [each coordinate space propagator is Fourier-transformed, with momentum conservation at each vertex, minus overall δ^4(external)]

and: $nV = N + 2P$ ⊛ each external particle ends at a vertex, each propagator at 2.

⊛ n lines at each vertex

⊗ Now: $D = dL - 2P$ — counting powers of momenta from d-dimensional loop integration and propagators to get superficial degree-of-divergence

[since ϕ propagator goes like $\frac{1}{k^2-\mu^2+i\varepsilon}$; recall $(\Box+\mu^2)\phi_{free}=0$]

$$\therefore\ D = d(P-V+1) - 2P = (d-2)\overbrace{\tfrac{1}{2}(nV-N)}^{P} - dV + d$$

$$\boxed{D = d + \left[n\frac{(d-2)}{2} - d\right]V - \left(\frac{d-2}{2}\right)N}$$

$\therefore$ In $\boxed{d=4}$ we must have $n \leq 4$ for a renormalizable theory (else $D\uparrow$ as $V\uparrow$)

(Actually $n=3$ is pathological, because $\phi \to -\infty$ to minimize the energy; and $n=2$ is just the mass term).

⊗ In $\boxed{d=2}$ there is no restriction on the interaction for a renormalizable theory and $\boxed{\phi > 4}$ is allowed!

⊗ Next, let's do the $\boxed{\text{dimensional analysis in d-dimensions}}$

$$\mathrm{Dim}(m_0\phi^2) = (\mathrm{mass})^d \Rightarrow \mathrm{Dim}(\phi) = (\mathrm{mass})^{\frac{d-1}{2}}$$

$$\therefore\ \mathrm{Dim}(\lambda_0\phi^n) = (\mathrm{mass})^d \Rightarrow \mathrm{Dim}(\lambda_0) = (\mathrm{mass})^{d-n\left(\frac{d-1}{2}\right)}$$

⊗ Now consider an **arbitrary amplitude** with **N external lines** (and any number of loops) (Lorentz invariant)

⊗ If we know the mass dimension of the amplitude, then we will know its degree of divergence

⊛ Well, an amplitude with N external lines would be generated at first order if we had

$$\mathcal{L}_{fake} = \eta \, \phi^N$$

∴ Have exactly N external particles in order to eat all contractions

i.e. $S^{(1)}_{fake} = -i\eta \int d^4x \, \langle f | \phi^N | i \rangle$

$\Rightarrow \mathcal{M}^{(1)}_{fake} =$  $= -i\eta \times (\#\text{ permutations})$

(Lorentz invariant)

⊛ So this uniquely fixes the dimension of the amplitude in the actual ϕ^n theory we're considering:

$$\mathrm{Dim}(\mathcal{M}_{N\ \text{particles}}) = \mathrm{Dim}(\eta) = (\text{mass})^{d - N\left(\frac{d-1}{2}\right)}$$

⊛ By dimensional analysis, the superficial degree of divergence D for a diagram with V vertices satisfies:

$$\lambda_0^V \Lambda^D = \mathrm{Dim}(\eta)$$

$$V\left[d - n\left(\frac{d-1}{2}\right)\right] + D = d - N\left(\frac{d-1}{2}\right)$$

in agreement with box at top of pg 143, based on "direct" analysis of loop integration

∴ We support the characterization we found with a less complete analysis of QEDd, on pg. 141