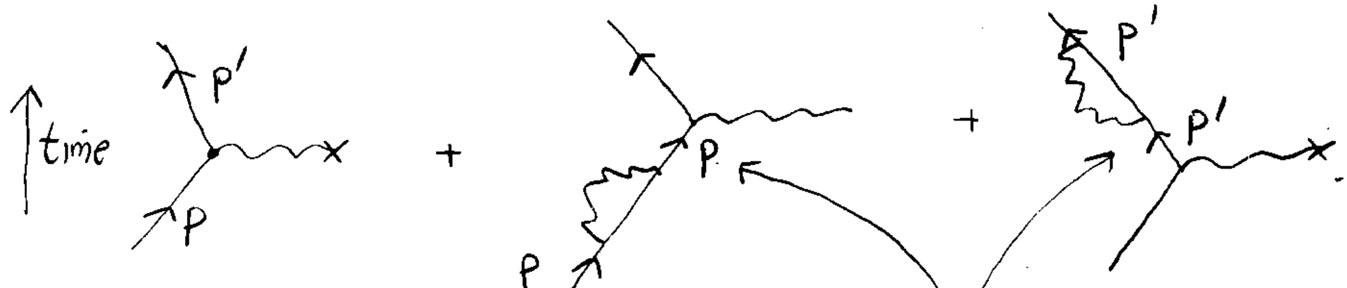
Ch. 7-1 $\boxed{\text{Electron Self-Energy}}$

We only do pgs. 216-222 for now

⊗ These diagrams have divergences which play <u>two</u> roles: one is to <u>exactly cancel</u> the vertex diagram divergence! and one is to "renormalize" the electron mass.

⊗ The first feature will seem accidental for now. But we'll see later that it's exact (Ward Takahashi Identity), and has deep consequences

⊛ To understand how to deal with the diagrams:



we already face a bizarre problem: according to our rules, there are intermediate e^- propagators with factors

e.g: $\dfrac{i(\not p+m)}{p^2-m^2+i\varepsilon} \sim \dfrac{i(\not p+m)}{\not p+i\varepsilon}$ since $p^2 = p'^2 = m^2$ $= \infty$!

⊛ The origin of this problem is that we assumed that we could switch off interactions of the e^- with the EM field in the remote past / future (we identified the external lines, say coming in from $t=-\infty$, as noninteracting, "free" electrons)

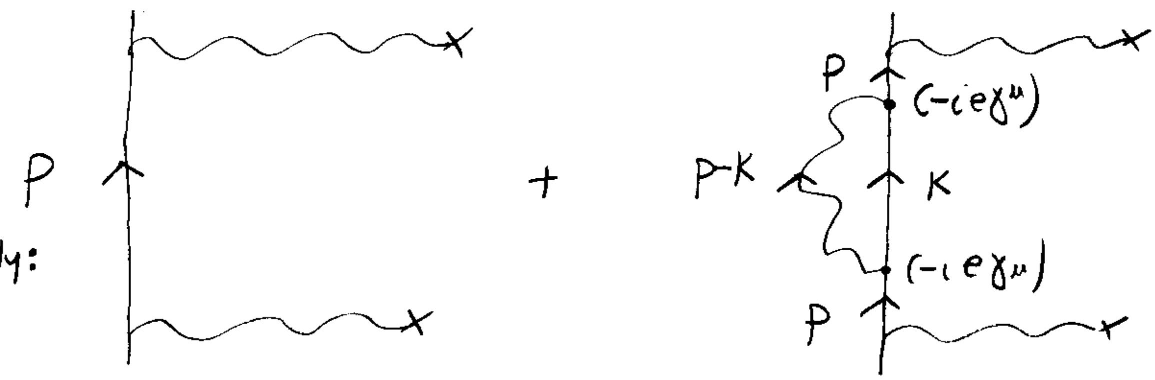
⊛ However, the "bare" electron represented by the external line interacts with itself at all times, including e.g the remote past.

⊛ We must find the correct way to identify the "asymptotic" (i.e. $t \to \pm\infty$) <u>isolated</u>, <u>yet</u> <u>self-interacting</u>, electron:

the LSZ reduction formalism

⊗ Will do this rigorously later, in Sects 7.1, 7.2. For now, we

⊛ First consider (instead of an external electron), an internal electron that couples to photons at very remote times in past / future:



Here, generally: $p^2 \neq m_0^2$

(the full amplitude includes also vertex diagrams, but forget those for now)

⊛ The electron lines give the factors:

$$\frac{i(\not p + m_0)}{p^2 - m_0^2 + i\varepsilon} + \frac{i(\not p + m_0)}{p^2 - m_0^2 + i\varepsilon}\left[-i\Sigma_2(p)\right]\frac{i(\not p + m_0)}{p^2 - m_0^2 + i\varepsilon}$$

sandwiched between the same photon vertices / propagators,

(just a convenient factor) odd

$$-i\Sigma_2(p) = (-ie)^2 \int \frac{d^4k}{(2\pi)^4}\, \gamma^\mu \frac{i(\not k + m_0)}{k^2 - m_0^2 + i\varepsilon}\, \gamma_\mu \frac{-i}{(p-k)^2 - \mu^2 + i\varepsilon}$$

where our use of m_0 will become clear soon enough.

⊛ Σ_2 has both of the diseases that we saw in the vertex function:

$$\Sigma_2 \underset{k_E}{\overset{\text{large}}{\sim}} \int^{\Lambda} k_E^3\, dk_E \frac{1}{k_E^4} \sim \ln \Lambda \rightarrow \infty$$

though the I.R. problem is manifested in a more subtle way!

⊛ Σ_2 <u>itself</u> is actually I.R. convergent:

small compared to

$$\Sigma_2 \underset{s_E = p-k}{\overset{\text{small}}{\sim}} \int_{s_{min}} s^3\, ds\, d\Omega_4 \frac{1}{s^2 - 2s\cdot p + p^2 - m_0^2} \frac{1}{s^2}\Bigg|_{\substack{\text{e.g.} \\ \text{at} \\ p^2 = m_0^2}} \sim \underbrace{\int_{s_{min}} \frac{ds\, s^3\, d\Omega_4}{(p\cdot s)(s^2)}}_{= \text{convergent}}$$

but to extract <u>physical</u> quantities we will <u>actually</u> require:

$$\frac{d}{dp}\Sigma_2(p^2)\Big|_{p^2 = m^2} \sim \int s^3\, ds \frac{1}{\ldots} \sim \ln s_{min} \rightarrow \infty.$$

- To regulate the IR divergence, we've introduced a "true" photon mass μ, as before. We'll also regulate U.V. with a fictitious heavy photon in a minute

II.2

- $\gamma^\mu (\not{k} + m_0)\gamma_\mu = -2\not{k} + 4m_0$ ← $(\ell + xp)$

- $$\frac{1}{k^2 - m_0^2 + i\varepsilon}\,\frac{1}{(p-k)^2 - \mu^2 + i\varepsilon} = \int_0^1 dx\,dy\,\delta(x+y-1)\,\frac{(2-1)!}{D^2}$$

$$D = y\left[k - m_0^2 + i\varepsilon\right] + x\left[(p-k)^2 - \mu^2 + i\varepsilon\right]$$

$$= k^2\underbrace{(x+y)}_{=1} - 2k\cdot xp + xp^2 - y m_0^2 - x\mu^2 + i\varepsilon$$

$$= (k - xp)^2 - x^2p^2 + xp^2 - \overbrace{(1-x)}^{y} m_0^2 - x\mu^2 + i\varepsilon$$

$$= \ell^2 - \Delta + i\varepsilon$$

where
$$\boxed{\begin{aligned}\ell &= k - xp \\ \Delta &= -x(1-x)p^2 + (1-x)m_0^2 + x\mu^2\end{aligned}}$$

No y dependence left, so can trivially do that parameter integral

- Notice that we haven't committed to $p^2 = m_0^2$
- Now shift integration: $k = \ell + xp$, drop terms linear in ℓ

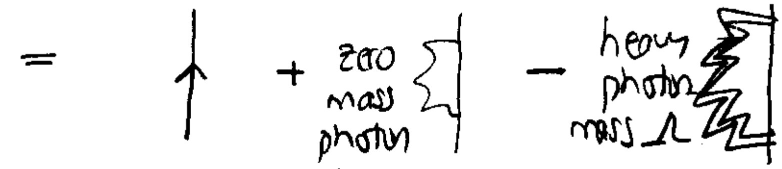
$$-i\Sigma_2^{REG}(p) = -e^2\int_0^1 dx\,(-2x\not{p} + 4m_0)$$

$$\times \int \frac{d^4\ell}{(2\pi)^4}\left[\frac{1}{(\ell^2 - \Delta + i\varepsilon)^2} - \frac{1}{(\ell^2 - \Delta_\Lambda + i\varepsilon)^2}\right] \equiv I_2$$

- Now we've subtracted the Pauli-Villars heavy photon regulator

$$\Delta_\Lambda = \Delta(\mu^2 \to \Lambda^2)$$

⊙ That is, we imagine total amplitude in ad-hoc modified QED



(with temporary I.R. regulator mass μ)

⊙ Notice the U.V. divergence in the shifted loop integral

$$\int d^4 \ell \; \frac{1}{(\ell^2 - \Delta + i\varepsilon)^2} \;\overset{\text{large}}{\underset{\ell_E}{\sim}}\; \int d\ell_E \frac{\ell_E^3}{\ell_E^4} \sim \ln \infty$$

which is mass independent, and is ∴ subtracted out by "heavy photon".

⊙ We can use exactly the same Wick Rotation technology (cf. pgs. 28-31, and pg. 36) as before:

$$I_2 = i\,(-1)^2 \frac{2\pi^2}{(2\pi)^4} \int_0^\infty d\ell_E \, \ell_E^3 \left[\frac{1}{(\ell_E^2 + \Delta)^2} - \frac{1}{(\ell_E^2 + \Delta_\Lambda)^2} \right]$$

$$x = \ell_E^2 \quad = i\,(-1)^2 \frac{2\pi^2}{(2\pi)^4} \frac{1}{2} \int_0^\infty dx \; x \left[\frac{1}{(x+\Delta)^2} - \frac{1}{(x+\Delta_\Lambda)^2} \right]$$

$x + \Delta - \Delta$ $\qquad$ $x + \Delta_\Lambda - \Delta_\Lambda$

cancel

$$= i\,(-1)^2 \frac{2\pi^2}{(2\pi)^4} \frac{1}{2} \int_0^\infty dx \left[\frac{1}{(x+\Delta)} - \frac{1}{(x+\Delta_\Lambda)} \right] \qquad \ln\left(\frac{x+\Delta}{x+\Delta_\Lambda}\right)\Bigg|_0^\infty$$

from bottom p. 41

$$\boxed{\Sigma_2^{REG}(p) = \frac{\alpha}{2\pi} \int_0^1 dx\,(2m_0 - x \not{p}) \log\left[\frac{\Delta_\Lambda}{\Delta}\right]}$$

P+S Eq. (7.19)

$$\Delta = -x(1-x)p^2 + (1-x)m_0^2 + x\mu^2$$

$$\Delta_\Lambda = \qquad \text{''} \qquad + x\Lambda^2$$

⊛ Now to understand the significance of Σ_2, we can proceed in one of two ways.

⊛ But first, notice the convenient identity (using $\not{p}\not{p} = p^2$):

⊛ $$\frac{\not{p}+m_0}{p^2 - m_0^2 + i\varepsilon} = \frac{\cancel{\not{p}+m_0}}{(\cancel{\not{p}+m_0} - i\tilde{\varepsilon})(\not{p} - m_0 + i\tilde{\varepsilon})} \equiv \frac{1}{\not{p} - m_0 + i\tilde{\varepsilon}}$$

($\tilde{\varepsilon} = \frac{1}{m_0}\varepsilon > 0$ all we care about)

⊛ So back to "total" propagator on p 40.

$$\frac{i}{\not{p}-m_0+i\varepsilon}$$  $+$ $i/\not{p}-m_0+i\varepsilon$ $-i\Sigma_2(p)$  $i/\not{p}-m_0+i\varepsilon$ $\equiv S_F^{tot}$

[I] Since $\Sigma_2 \sim \alpha \ll 1$, let's approximate the two terms as a geometric series:

$$S_F^{tot} = \frac{i}{\not{p}-m_0+i\varepsilon} + \frac{i}{\not{p}-m_0+i\varepsilon}(-i\Sigma_2)\frac{i}{\not{p}-m_0+i\varepsilon}$$

($\underbrace{}_{}$4x4 matrix (so watch order, but in principle, show anyway it commutes with $\not{p}$)

$$\approx \frac{i}{\not{p} - m_0 - \Sigma_2(p) + i\varepsilon} + O(\alpha^2)$$ since:

[Identify: $A = -i(\not{p}-m_0+i\varepsilon)$
$B = -i\Sigma_2$]

$\boxed{\text{Identity}}$: $$\frac{1}{A-B} = \frac{1}{A} + \frac{1}{A}B\frac{1}{A} + \frac{1}{A}B\frac{1}{A}B\frac{1}{A} + \cdots$$

(for operators A and B not necessarily commuting).

Proof: left multiply by $(A-B)$,

$$1 = (A-B)A^{-1} + (A-B)A^{-1}BA^{-1} + (A-B)A^{-1}BA^{-1}BA^{-1} + \cdots$$

$$= 1 - \cancel{B}A^{-1} + (\cancel{B}A^{-1} - BA^{-1}\cancel{BA^{-1}}) + (BA^{-1}\cancel{B}A^{-1} - BA^{-1}BA^{-1}BA^{-1}) + \cdots$$

$\boxed{\text{QED}}$

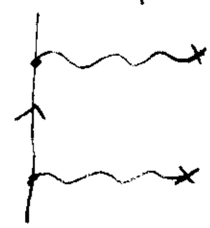
[II] In fact, the entire geometric series corresponds to an infinite series of actual Feynman diagrams, which comprise a subset of the complete amplitude

$$\frac{1}{A} + \frac{1}{A} B \frac{1}{A} + \frac{1}{A} B \frac{1}{A} B \frac{1}{A} + \cdots$$

⊗ This partial summation of an infinite subset of Feynman diagrams will be "justified" later as a consistent scheme for organizing the set of all diagrams

$\boxed{\text{Now to analyze significance of } \Sigma_2(p)}$

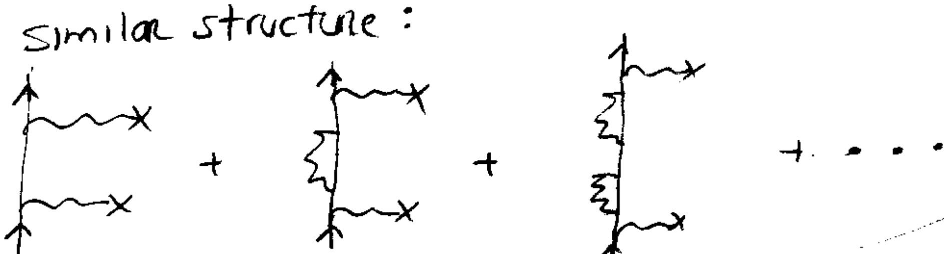
⊗ Before turning on electron self-interaction, the electron propagates with a factor of:



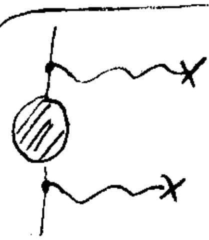
$$= \frac{i}{\not p - m_0 + i\varepsilon} = \frac{i(\not p + m_0)}{p^2 - m_0^2 + i\varepsilon}$$

which blows up when the 4-momentum satisfies the on-shell condition $p^2 = m_0^2 = (\text{electron mass})^2$. It actually has two simple poles, at $p^0 = \pm E(\vec{p})$.

⊗ The effect of self-interactions is to produce a factor with a similar structure:



$$\equiv$$



$$\equiv \frac{i}{\not p - m_0 - \Sigma(\not p) + i\varepsilon}$$

⊗ The notation here is that the bulb and lines  are all represented by this one factor (i.e. no additional factors for the lines)

⊗ This "dressed" propagator will blow up, but $\boxed{\text{NOT}}$ at the "free" or $\boxed{\text{BARE}}$ electron mass m_0, but at a $\boxed{\text{RENORMALIZED}}$ mass, which includes the effects of self-interactions

(*) Because we can __never__ turn off the electron's self-interactions, the parameter m_0 is __not__ a physical quantity.

(*) It is the pole of the "__dressed__" propagator that identifies the __physical__ ("renormalized") electron mass.

(*) This argument is a bit hand-wavy. It will be justified rigorously later, when we come back to Sects. 7.1 + 7.2.

To identify the physical mass, let's find the location of the (two) simple poles in the "dressed" propagator

(*) Since Σ is a 4×4 matrix, let's define:

$$\boxed{\Sigma(p) = A(p^2)\,\not{p} + B(p^2)\,I}$$

understood, as always cf. bottom p.42.

Then: $\boxed{S_{tot}} = \dfrac{i}{\not{p} - m_0 - \Sigma(p) + i\varepsilon}$

$$= \frac{i}{(1-A(p^2))\,\not{p} - (m_0 + B(p^2)) + i\varepsilon}$$

$$= \frac{i}{[(1-A)\,\not{p} - (m_0+B) + i\varepsilon]}\;\frac{[(1-A)\,\not{p} + (m_0+B) - i\varepsilon]}{[(1-A)\,\not{p} + (m_0+B) - i\varepsilon]}$$

can drop

$$\boxed{= \frac{i\,[(1-A)\,\not{p} + (m_0+B)]}{(1-A)^2 p^2 - (m_0+B)^2 - i\tilde{\varepsilon}}}$$

$\tilde{\varepsilon} = (m_0 + B)\,\varepsilon \underbrace{> 0}_{\text{all we care about}}$ if $B > -m_0$.

(*) Denominator has a pair of simple poles at

$$p^0 = \pm\sqrt{\vec{p}^{\,2} + m_{physical}^2}$$

i.e. at $p^2 = m_{physical}^2 \equiv m^2$, where:

$$[1-A(m^2)]^2 m^2 = [m_0 + B(m^2)]^2$$

$$\Rightarrow [1-A(m^2)]\, m = [m_0 + B(m^2)]$$

$$\therefore \quad m - m_0 = A(m^2)\, m + B(m^2)$$

(*) Notice that we end up with a compact expression in terms of $\Sigma(\not{p} \equiv m) = A(p^2=m^2) \times (\not{p} \equiv m) + B(p^2=m^2)$

$$\therefore \boxed{\delta m \equiv m_{physical} - m_0 = \Sigma(\not{p} \equiv m_{physical})}$$

(*) We will evaluate this explicitly on pg 51

(*) The electron self-interactions also modify the amplitude for creating a "true" electron from the vacuum, with the quantum operator Ψ_{op}

cf. $\Psi_{op}(x)|0\rangle \underset{\text{theory}}{\overset{\text{free}}{=}}$ "1" $\cdot$ $\left|\begin{matrix}\text{electron sharply}\\ \text{localized at } x\end{matrix}\right\rangle$

(*) What about when we turn on self-interaction?

$\boxed{\text{Go back to Denominator of } S_{tot}\text{, bottom last page:}}$

$$D(p^2) \equiv [1-A(p^2)]^2 p^2 - [m_0 + B(p^2)]^2$$

$$= 0 \text{ at } p^2 \equiv m^2, \text{ as above.}$$

⊛ Near the pole, we can Taylor expand:

$$D(p^2 \approx m^2) \approx \underbrace{D(p^2 = m^2)}_{\equiv 0} + \frac{dD}{dp^2}\Big|_{p^2=m^2}(p^2 - m^2) + O((p^2-m^2)^2)$$

⊛ $$\frac{dD}{dp^2}\Big|_{m^2} = (1 - A(m^2))^2 - 2\frac{dA}{dp^2}\Big|_{m^2}(1 - A(m^2))\, m^2$$

$$- 2\frac{dB}{dp^2}\Big|_{m^2}\underbrace{(m_0 + B(m^2))}_{= m(1 - A(m^2))}$$

by top of last page

$$\therefore \frac{\partial D}{\partial p^2}\Big|_{m^2} = [1 - A(m^2)]\Big[\underbrace{1 - A(m^2) - 2m\Big(\frac{dA}{dp^2}\, m + \frac{dB}{dp^2}\Big)}\Big]$$

⊛ The last bracket can be written in a compact way:

$$\Sigma(p) = A(p^2)\, \not{p} + B(p^2)$$

and define:

$$\boxed{\frac{d}{d\not{p}}\Sigma(p) \equiv \frac{\partial}{\partial \not{p}}\Sigma(p) + \frac{dp^2}{d\not{p}}\frac{\partial}{\partial p^2}\Sigma(p^2)}$$

where: $p^2 \equiv \not{p}^2$

$$\therefore \frac{d}{d\not{p}}\Sigma(p)\Big|_{\not{p}\equiv m} = A(m^2) + 2m\frac{\partial}{\partial p^2}(A(p^2)\, m + B(p^2))\Big|_{p^2=m^2}$$

$$\therefore \frac{\partial D}{\partial p^2}\Big|_{m^2} \equiv [1 - A(m^2)]\Big[1 - \frac{d\Sigma}{d\not{p}}\Big|_{\not{p}=m}\Big]$$

$$\therefore\ S_{tot}\Big|_{p^2 \approx m^2} \approx \frac{i\left[(1-A(m^2))\,\not p + \underbrace{(m_0 + B(m^2))}\right]}{D(p^2 \approx m^2) - i\tilde{\varepsilon}}$$

pg. 46 from top 1

⊛ again use $= (1 - A(m^2))m$

$$\approx \frac{i\,\cancel{[1-A(m^2)]}\,[\not p + m]}{\cancel{[1-A(m^2)]}\left[1 - \frac{d\Sigma}{d\not p}\Big|_{\not p = m}\right](p^2 - m^2 - i\tilde{\tilde{\varepsilon}})}$$

Hopefully prefactors still positive. Should be, since $A, \Sigma, B \sim O(\alpha)$, small compared to 1.

$(\not p + m + i\varepsilon)(\not p - m - i\varepsilon)$

$$\therefore\ \boxed{S_{tot} \approx \frac{i\,Z_2}{\not p - m - i\varepsilon} + \text{terms regular at } p^2 = m^2}$$

where:

$$\boxed{\begin{aligned} Z_2^{-1} &\equiv \left(1 - \frac{d}{d\not p}\Sigma(\not p)\Big|_{\not p = m}\right) \\ \delta m = m - m_0 &= \Sigma(\not p = m) \end{aligned}}$$

⊗ We "understand" the renormalization of the mass.

⊛ What is the significance of the Z_2 factor?

⊛ Justified by our detailed derivation.

Side Bar: "Quick" derivation of these factors

⊛ Assuming it's legitimate to define $p^2 = \not p^2$ in calculating derivative.

$$S_{tot} = \frac{i}{\not p - m_0 - \Sigma(\not p)} = \frac{i}{0 + \frac{d}{d\not p}(\not p - m_0 - \Sigma(\not p))\Big|_{\not p = m}(\not p - m) + O(\not p - m)^2}$$

where $[\not p - m_0 - \Sigma(\not p)]_{\not p \equiv m} = 0$ and $Z_2^{-1} = \left[1 - \frac{d}{d\not p}\Sigma(\not p)\right]_{\not p \equiv m}$

Significance of Z_2; and: What about *external* lines?

(mathematically, the residue of the poles of the interacting $\bar{e}$ propagator)

⊛ The electron field operator in the fully self-interacting theory does *not* *only* create/destroy isolated (self-interacting) electrons

⊛ Compare with the *free* theory where "yes":

$$\bar{\psi}^{\text{free}}_{op}(x)\,|0\rangle = \text{"}1\text{"} \times \left|\begin{matrix}\text{free electron localized at } x, \\ \text{of mass } m_0\end{matrix}\right\rangle$$

⊛ More precisely:

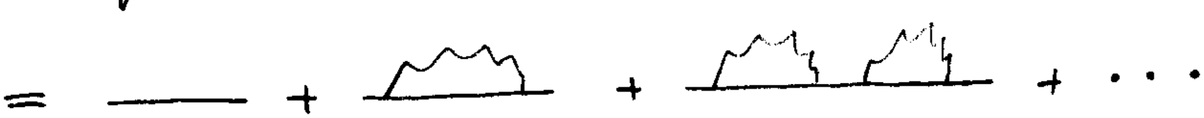
$$\langle 0|T\left(\psi^{\text{free}}_{op}(y)\,\bar{\psi}^{\text{free}}_{op}(x)\right)|0\rangle \overset{S_F^{\text{free}}(x-y)}{\equiv} \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-y)}\, \frac{i \times 1}{\not{p} - m_0 + i\varepsilon}.$$

⊛ Although it's not obvious yet, what we did was to calculate the Feynman diagrams for the *fully interacting* "two-point function" in momentum space (i.e. Fourier transformed):
(it's more than just an "electron propagator" now!!)

$$⊛ \int d^4x\, e^{ipx} \left\langle 0|T\left(\psi^{\text{full}}_{op}(x)\,\bar{\psi}^{\text{full}}_{op}(0)\right)\right\rangle$$

can use Wick's theorem to express in terms of Feynman diagrams

$$= \frac{i\,Z_2}{\not{p} - m - i\varepsilon} + (\text{terms regular at } p^2 = m^2)$$



⊛ Conclusion:

$$\bar{\psi}^{\text{full}}_{op}(x)\,|0\rangle = \sqrt{Z_2}\,\left|\begin{matrix}1 \text{ fully self-interacting} \\ \text{electron of mass } m\end{matrix}\right\rangle$$

$$+ \text{ other states!}$$

⊗ The reason for the $\sqrt{\ }$ is that the propagator involves two quantum fields (this is fairly hand-wavy; the argument below for external lines even more so: all is made rigorous by LSZ!)

⊗ The other states that are created/destroyed by $\bar{\psi}^{\text{full}}$ all have charge 1e; e.g: $e^-\gamma\gamma\ldots$, $e^- e^+ e^- \gamma\gamma\gamma\ldots$, etc.

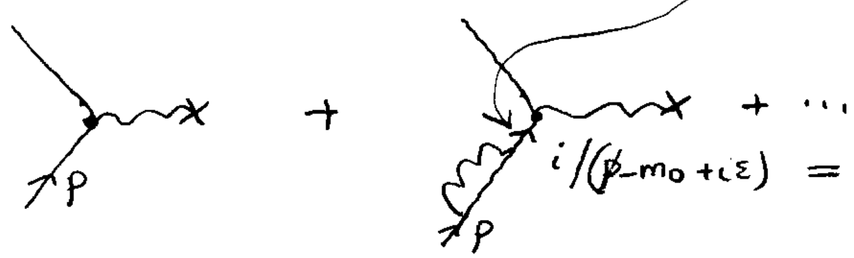
$$\boxed{\text{What about external electron lines ?}}$$

⊗ Self energy corrections on external lines again serve to renormalize the electron mass, and to tell us that

$$\underbrace{\bar{\psi}^{\text{full}}(x) \mid \bar{e}(p)\rangle}_{} = \sqrt{Z_2}\left(\frac{m}{E(p)V}\right)^{1/2} u(p)\, e^{ip\cdot x}$$

i.e. the quantum field does more than create 1 "true" electron

⊗ The bizarre "vanishing propagator"



$i/(\not{p} - m_0 + i\varepsilon) = 0$ at $\not{p} = m_0$

tells us we haven't yet properly summed electron self-interactions on the external line (though we've done it for "internal" electron propagators)

⊗ We need a procedure for identifying asymptotic states for input as external lines into Feynman diagrams.

⊗ It's basically the same as identifying the pole and residue in the interacting "electron propagator"

$$\boxed{\text{Upshot of LSZ "reduction-formula}}$$

makes sense: these just tell us to replace $m_0 \to m$ in e.g. $u(p)$ etc...

(1) Disregard all self-energy corrections to external lines

(2) Multiply amplitude by $\sqrt{Z_2}$ for each external line

cf. external state contraction

Finally: Explicit evolution of δm (and Z_2)

- $\delta m \equiv m - m_0 = \Sigma'(\not{p}=m)$ by pg. 48

$$= \frac{\alpha}{2\pi}\int_0^1 dx\,(2m_0 - mx)\log\left(\frac{\Delta_\Lambda}{\Delta}\right) \quad \text{from pg. 42}$$

$$\Delta = -x(1-x)p^2 + (1-x)m_0^2 + x\mu^2 \qquad \underline{\underline{\text{at}}}\ p^2 = m^2$$
$$\Delta_\Lambda = \quad '' \qquad\qquad\qquad \Lambda^2$$

- We don't yet know the value of m, required as input to RHS
- However $m = m_0 + O(\alpha)$, and the RHS already has this overall factor of α. Since we haven't calculated the $O(\alpha^2)$ perturbative ("two-loop") correction to Σ', we can freely set $m = m_0$ on the RHS, for consistent calculation to this order!
- On the other hand $m = m_0 + O(\alpha)\ln\Lambda$, due to the U.V. divergence in Σ'_2, so is this counting of "small" corrections valid?
- Notice that δm is $\underline{\text{I.R. finite}}$, because a logarithmic singularity is integrable (recall our power counting on pg. 40: Z_2 $\underline{\text{is}}$ IR divergent because it comes from $d/dp^2\, \Sigma'$).
- All physical quantities are finite as $\mu \to 0$ (see Sect. 6.4). (δm not physical, but anyway...) Here, we can apparently "freely" set $\mu^2 = 0$

$$\therefore\ \delta m \cong \frac{\alpha}{2\pi} m_0 \int_0^1 dx\,(2-x)\log\left[\frac{(1-x)^2 m_0^2 + x\Lambda^2}{(1-x)^2 m_0^2}\right]$$

$$= \frac{\alpha}{2\pi} m_0 \int_0^1 dx\,(2-x)\left[\log\left(\frac{\Lambda^2}{m_0^2}\right) + \log\left(\frac{x}{(1-x)^2} + \frac{m_0^2}{\Lambda^2}\right)\right]$$

⊛ So, let's evaluate the mass shift in the $\Lambda \to \infty$ limit, dropping terms that vanish in this limit.

⊛ Only relevant limit for "true" QED

$$\therefore \quad \delta m \xrightarrow[\Lambda\to\infty]{} \frac{\alpha m_0}{2\pi} \log\left(\frac{\Lambda^2}{m_0^2}\right) \int_0^1 dx\,(2-x)$$

$$+ \frac{\alpha m_0}{2\pi} \int_0^1 dx\,(2-x)\left[\log x - 2\log(1-x)\right]$$

⊛ The second line is negligible compared to the first, but I'll evaluate it anyway.

$$\therefore \int_0^1 (2-x)\log x\,dx - 2\int_0^1 dy\,(1+y)\log y \quad (y = 1-x)$$

$$= (-1)\left\{2 - \tfrac{1}{4} - 2\left(1 + \tfrac{1}{4}\right)\right\} = +3/4$$

⊛ For future reference, here is a neat identity:

$$\boxed{\int_0^1 dx\, x^n \log x = -\frac{1}{(n+1)^2}}$$ for all $n \geq 0$.

$$\text{Proof: } = \frac{1}{n+1}\int_0^1 d(x^{n+1})\log x = \underbrace{\frac{1}{n+1}x^{n+1}\log x\Big|_0^1}_{=0} - \frac{1}{n+1}\int_0^1 x^{n+1}\frac{1}{x}\,dx$$

$$\therefore \quad \boxed{m = m_0 + \frac{3\alpha}{4\pi}\left[\log\left(\frac{\Lambda^2}{m_0^2}\right) + \frac{1}{2}\right]}$$

⊛ This part will depend on details of regularization scheme, but not the coefficient of the logarithm!

log piece in P+S (7.29)

full expression in Itzykson + Zuber (7-42)

⊛ So how to interpret this result?

⊛ We hope that the complete, final, "theory of everything" (T.O.E.!) has no divergences. We can even imagine that we will be able to predict everything (including fundamental constants like m_0) from first principles in TOE.

⊛ The $\log\Lambda$ above expresses our ignorance of TOE physics at ultra-high energies $\therefore$ In QFT we cannot predict fundamental constants like the physical electron mass m.

⊛ But that's OK. We've now expressed the physical electron propagator entirely in terms of the physical mass!

⊗ Since m_0 has no physical significance (so far!!) in QFT, we should go back to all of our successful tree-level calculations and replace $m_0 = m_{physical} - O(\alpha)$, and only then substitute $m_{physical}\, c^2 = 0.511$ MeV

⊗ On the other hand, this shift should be disregarded until you calculate the next order correction, also of $O(\alpha)$, to your process.

⊗ Whoa! Isn't the shift actually infinite? No! You have no business taking QFT at face value in the limit $\Lambda \to \infty$, because QFT certainly fails at some huge (but finite) Λ.

⊗ Example: Suppose that QED is valid up to the energy scale at which space-time (gravity) itself is subject to quantum fluctuations

$$\Lambda_{Planck} \equiv \sqrt{\frac{\hbar c}{G_{Newton}}} \sim 10^{19}\ \text{GeV}$$

(the only energy scale that can be formed from the fundamental constants of gravitation, quantum mechanics, and relativity)

$$\therefore\ \frac{\delta m}{m}\bigg|_{Planck} \sim \frac{3\alpha}{4\pi} \times 2 \times \log\left(\frac{10^{19}\ \text{GeV}}{0.511\ \text{MeV}}\right) \sim 15\%$$

⊗ "Decoupling Theorem": QFT analogue of Newton's 3RD law!:

⊗ The point is that the divergences in QFT are almost always extremely weak (logarithmic in Λ) and so we are extremely insensitive to the unknown physics

⊗ Moreover: in a consistent calculation, we will never "see" the shift between m_0 and m. At every order in perturbation theory, only the physical mass m appears, as in our calculation of S_{tot} (the "all orders" not obvious yet: see Ch. 10).

$S_{tot} = iZ_2/(\not{p} - m - i\epsilon)$

Divergences + Renormalization in <u>Classical</u> Electrodynamics too!

⊗ In some sense, the problem is even <u>worse</u> in the classical theory!

⊗ The total energy in the Coulomb field of a point charge is infinite, so suppose instead that the electron is a spherical <u>shell</u> of radius a

(i.e. we impose a U.V. cutoff on the unknown structure of the electron):

$$\therefore\ E^{REG}_{\text{Coulomb}} = \frac{1}{2}\int_a^\infty \left(\frac{e^2}{4\pi r^2}\right)^2 d^3\vec{r} = \frac{\alpha}{2a} \equiv \frac{1}{2}\alpha\Lambda$$

in terms of an energy cutoff $\Lambda = \hbar c/a \equiv 1/a$.

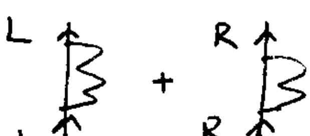
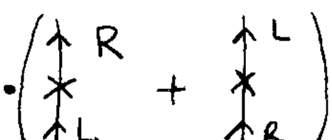
⊗ The classical divergence is <u>linear</u> instead of logarithmic!

⊗ This is already an excellent example of how <u>unknown</u> U.V. physics can "soften" the deficiencies of an incomplete theory.

⊗ Recall Mandl + Shaw, Problem 4.5:

$$\mathcal{L}_{QED} = i\bar{\Psi}_L \not{D}\Psi_L + i\bar{\Psi}_R \not{D}\Psi_R + m_0\left(\bar{\Psi}_R\Psi_L + \bar{\Psi}_L\Psi_R\right)$$

⊗ So m_0 couples Ψ_R to Ψ_L. If $m_0=0$, they <u>stay</u> uncoupled!

i.e.: $\Sigma(m_0=0) =$  $\neq \delta m \cdot \Big($  $\Big)$

⊗ In fact, with $m_0=0$:  $= 0$! (Try it!)

$\therefore$ δm must vanish with m_0, so by dimensional analysis the <u>only</u> divergent dependence on Λ allowed is: $\delta m \propto m_0 \log(\Lambda/m_0)$!

Preview of Ward-Takahashi Identity

⊗ The ultra-violet divergence in δm is absorbed by expressing the propagator in terms of m_{physical}:

$$S_{\text{tot}} = \frac{i\, Z_2}{\not{p} - m_{\text{physical}} + i\varepsilon} + \text{terms regular at } p^2 = m^2$$

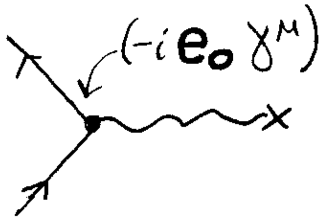
[SIDE-BAR: The self-energy graph has more physics, because of !. This means that it's more than just a photo-copy of the free propagator, when $p^2 \neq m^2$:

$$S_{\text{tot}} = \left.\frac{i\, Z_2}{\not{p} - m - \Sigma_R(p) + i\varepsilon}\right\} \text{ where } \underset{\textstyle \Sigma_R(p^2 \simeq m^2)}{\overset{\textstyle O((p^2-m^2)^2)}{\parallel}}$$

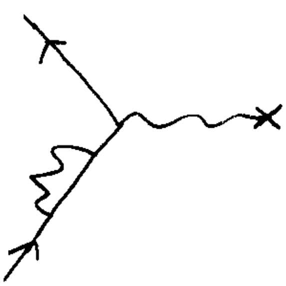
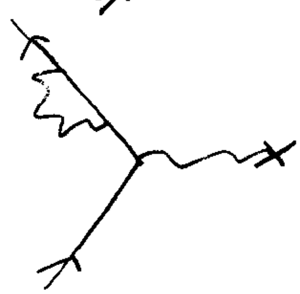
and this contributes to processes e.g ]

⊛ The U.V. divergence in Z_2 is also absorbed by expressing e.g. cross-sections in terms of the physical electron charge!

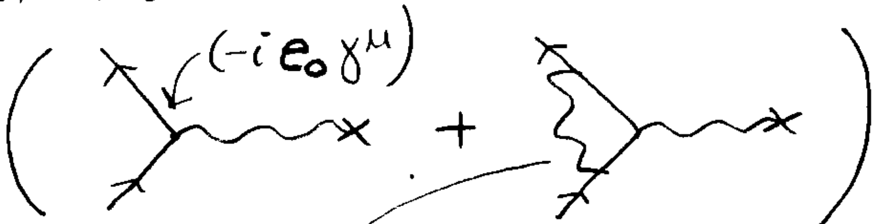
⊗ Go back to scattering amplitude:

$\mathcal{M} =$ 

$+$ 

$+$  $+$ 

⊗ According to our hand-wavy argument on pg. 50, LSZ say "drop graphs with self-energy insertions on external lines, and multiply remaining graphs by $\sqrt{Z_2}$ for each external line (and replace $m_0 \to m$)"

$\therefore \mathcal{M} = Z_2$ (


)

two external lines

[we are still neglecting  : get to it later]

⊛ all divergences were in δF_1.

$$= Z_2(-ie_0)\left[\left(1+\delta F_1^{REG}(q^2)\right)\gamma^\mu + i\,\delta F_2(q^2)\,\frac{\sigma^{\mu\nu}q_\nu}{2m}\right]$$

⊗ This is just the structure that we used for most general analysis of scatterings by electric/magnetic fields

$$\equiv -i\,e_{physical}\left[F_1^{physical}(q^2)\,\gamma^\mu + i\,F_2^{physical}(q^2)\,\frac{\sigma^{\mu\nu}q_\nu}{2m}\right]$$

where:

⊛ $F_{1,2}^{physical}(q^2) = F_{1,2}^{REG}(q^2) \,/\, F_{\underline{\underline{1}}}^{REG}(q^2=0)$

⊛ $e_{physical} = e_0\, Z_2\, F_1^{REG}(q^2=0)$

[Recall $e_{physical} = e_0\, F_1(q^2=0)$, pg. 12;

the Z_2 factor is merely due to our separation of total amplitude into self-energy plus vertex parts]

⊗ It is conventional to call

$$Z_1^{-1} \equiv F_1^{REG}(q^2 = 0)$$

so that : $\Gamma_\mu(q \to 0) \to Z_1^{-1} \gamma^\mu$

$\therefore$ $\boxed{e_{physical} = e_0 Z_2 Z_1^{-1}}$ ⊗ Well: also $\times Z_3$ from $\Sigma_{\mu\nu}$: see pg. 61

⊗ Remarkably : $\boxed{Z_1 = Z_2 \text{ in QED}}$!

(See Assignment 2, P+S Problem 7.3)

⊗ This is not necessary for expressing physical cross-sections in terms of finite, measurable, physical constants (though it does have deep phenomenological consequences)

⊗ It's that relation which in general defines $e_{physical}$: it cannot be predicted theoretically, only input from experiment (and e_0 is just a parameter!)

⊗ Also, $\boxed{F_1^{physical} \text{ is finite}}$ since the divergence is independent of q^2:

(and since $F_2 = O(\alpha)$ at leading order, this renormalization doesn't change $a \cong \alpha/2\pi$).

⊗ $F_1^{physical}(q^2) = [1 + O(\alpha) \ln \Lambda + O(\alpha) f(q^2)] / [1 + O(\alpha) \ln \Lambda]$

$= 1 + O(\alpha) f(q^2) + O(\alpha^2)$, $f(q^2) = \text{finite}$.

$$\boxed{\begin{aligned} Z_1^{-1} &\equiv F_1^{REG}(q^2=0) \\ &= 1 + \frac{\alpha}{2\pi} \int_0 dx\,dy\,dz\,\delta(x+y+z-1)\left[\log\left(\frac{z\Lambda^2}{m^2(1-z)^2}\right) + \frac{m^2(1-4z+z^2)}{m^2(1-z)^2 + \mu^2 z}\right] \end{aligned}}$$

(pg. 37; P+S (7.32))

⊗ We've dropped μ^2 in logarithm, and Δ in numerator of logarithm, just as in calculation of δm (e.g. the terms with Λ^2 in numerator vanish as $\Lambda \to \infty$).

⊛ To the order that we have worked

$$Z_1^{-1} \equiv 1 + \delta F_1^{REG}(q^2=0) \cong 1 + O(\alpha)$$

$$\therefore \quad Z_1 \approx 1 - \delta F_1^{REG}(q^2=0) + O(\alpha^2)$$

$$\equiv 1 + \delta Z_1 .$$

⊛ Also $Z_2 = 1 + \delta Z_2 = 1 + O(\alpha)$,

so we want to prove :

$$\delta Z_1 = \delta Z_2 \text{ to } O(\alpha)$$

or $\boxed{\delta Z_2 + \delta F_1^{REG}(q^2=0) = 0}$ cf P+S p. 222, 229, 230.

⊛ From pgs. 48, 42 :

$$Z_2^{-1} = \left(1 - \frac{d}{d\not{p}} \Sigma(\not{p})\Big|_{\not{p}=m}\right)$$

$$\therefore \quad \delta Z_2 \cong \frac{d}{d\not{p}} \Sigma(\not{p})\Big|_{\not{p}=m} + O(\alpha^2)$$

$$= \frac{\partial}{\partial \not{p}} \Sigma(\not{p}) + 2m \frac{\partial}{\partial p^2} \Sigma(\not{p})$$

⊛ $$\Sigma(\not{p}) = \frac{\alpha}{2\pi} \int_0^1 dz\,(2m - z\not{p}) \log\left[\frac{z\Lambda^2}{-z(1-z)p^2 + (1-z)m^2 + z\mu^2}\right]$$

⊗ Again, we drop terms accompanying $z\Lambda^2$ in numerator, but now not μ^2, since we must differentiate:

$$\therefore \delta Z_2 = \frac{\alpha}{2\pi}\int_0^1 dz\left[-z\log\left(\frac{z\Lambda^2}{(1-z)^2m^2+\cancel{z\mu^2}}\right)\right.$$ $\to \phi$

cf. P+S (7.31)

$$\left.+\,2(2-z)\frac{z(1-z)\,m^2}{(1-z)^2m^2+z\mu^2}\right]$$

$\frac{\partial}{\partial \not p}$ ← $2m\,\partial/\partial p^2$

⊗ Now let's add $\delta F_1(q^2=0)$, from bottom pg. 57

$\left(\text{use: } \int_0^1 dx\,dy\,dz\,\delta(x+y+z-1) = \int_0^1 dz\int_0^{1-z} dy = \int_0^1 dz\,(1-z)\right)$:

$$\delta Z_2 - \delta Z_1 = \delta Z_2 + \delta F_1(q^2=0)$$

$$= \frac{\alpha}{2\pi}\int_0^1 dz\,(1-2z)\log\left(\frac{z\Lambda^2}{(1-z)^2m^2}\right)$$

$$+\,\frac{\alpha}{2\pi}m^2\int_0^1 dz\,(1-\cancel{z})\left[\frac{(1-4z+z^2)+2(2-z)z}{\cancel{(1-z)^2}m^2+z\mu^2}\right]$$

$\to 1-z^2 = (1+z)(\cancel{1-z})$

can now take $\to 0$

⊗ Second integral now I.R. finite

⊗ Notice also that the divergent term is

$$\sim \int_0^1 dz\,(1-2z)\times\log\left(\frac{\Lambda^2}{m^2}\right) = 0\times\log\frac{\Lambda^2}{m^2}\ !$$

(all divergences must cancel if $\delta Z_2+\delta F_1=0$ of course!)

⊗ The <u>finite</u> pieces <u>also</u> cancel:

$$\delta Z_2 - \delta Z_1 = \frac{\alpha}{2\pi}\int_0^1 dz\,\underbrace{(1-2z)}_{\to\,(2(1-z)-1)}\left[\log z - 2\log(1-z)\right]$$

$$+\ \frac{\alpha}{2\pi}\int_0^1 dz\,(1+z)$$

⊗ Use our identity from pg. 52: $\int_0^1 dz\, z^n \log z = -\frac{1}{(n+1)^2}$

$$\ddot{\circ} = \frac{\alpha}{2\pi}(-1)\left\{\overset{\frac{1}{2}}{1} \overset{-2\times(-\frac{1}{2})}{-2\times\frac{1}{4}} - 2\times\left(\frac{2}{4}-1\right)\right\} + \frac{\alpha}{2\pi}\times\frac{3}{2} = 0!$$

("proof" on P+S pg. 222 has many typos).

$$\boxed{\ddot{\circ}\ \text{Yes!}\ Z_1 = Z_2}$$

at least to $O(\alpha)$. All order proof to come, Sect. 7.4.

Deep phenomenological consequences of W.T. Identity

⊗ In a general QFT, we will have renormalized coupling: $g = Z_1^{-1} Z_2\, g_0$ ← bare coupling (cf p. 57)

but we won't have $Z_1 = Z_2$ (this occurs <u>only</u> because of a symmetry which <u>preserves</u> it: gauge symmetry)

⊗ Now, as we have seen $Z_{1,2} = Z_{1,2}(\text{particle mass})$.

⊗ Suppose we calculate $e_{\text{renormalized}}$ for all the different charged particles in the universe. Since they all have vastly different masses, we expect:

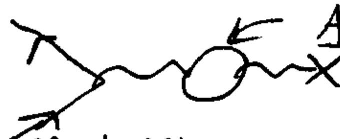
$$e_{\text{renormalized}}(\text{particle } 1) \neq e_{\text{renormalized}}(\text{particle } 2)$$

$\boxed{\text{unless}}$ Z_1 <u>exactly</u> equals Z_2

⊗ This relationship being "protected" by gauge symmetry is the only way to "explain" the experimental facts of:

e.g. $|Q_{proton} + Q_{electron}| / e < 10^{-21}$

i.e. we can't predict the charge, but thanks to W.T. identity, but we can predict that the charges of all particles undergo a universal, mass-independent renormalization factor

[That comes from  ← All quantum particles go in here, equally for all $m_{external}$.

$m_{external}$

the only piece we haven't calculated yet]

⊗ So if particles have equal e_0 in the underlying T.O.E, they will have equal e renormalized "in low energy QED"!

Explicit evolution of Z_2

(and P+S 7.31)

⊗ From top pg. 59:

$$\frac{2\pi}{\alpha}\,\delta Z_2 = \int_0^1 dz\,\underbrace{(-z)}_{-\frac{1}{2}}\log\left(\frac{\Lambda^2}{m^2}\right) - \underbrace{\int_0^1 dz\, z\left(\log z - 2\log(1-z)\right)}_{(-1)\left\{\frac{1}{4} - 2\times\left(-\frac{1}{4}+1\right)\right\} = +\frac{5}{4}}$$

$\to (1-z)+1$

$$+ 2m^2\int_0^1 dz\,(2-z)(1-z)\underbrace{\left[\frac{z}{(1-z)^2 m^2 + z\mu^2} - \frac{1}{(1-z)^2 m^2 + \mu^2}\right]}$$

$z \to 1$

$$+ 2m^2\int_0^1 dz\,(2-z)(1-z)\,\frac{1}{(1-z)^2 m^2 + \mu^2}$$

⊗ I subtracted the I.R. divergence ($z \to 1$) so it's finite, so rationalize, and then set $\mu^2 = 0$

$$\therefore \frac{2\pi}{\alpha}\,\delta Z_2 = -\frac{1}{2}\log\frac{\Lambda^2}{m^2} - \frac{5}{4}$$

$$-2\cancel{m^2}\int_0^1 dz\,(2-z)(1-z)\,\frac{(1-z)^3\cancel{m^2}}{[\cancel{m^2}(1-z)^2]^2}$$ ← from rationalizing the two denominators

$$+2m^2\int_0^1 dz\,(1-z)^2\,\frac{1}{(1-z)^2m^2+\cancel{\mu^2}}\nearrow 0 \quad + 2m^2\int_0^1 dz\,(1-z)\,\frac{1}{(1-z)^2m^2+\mu^2}.$$

$$= -\frac{1}{2}\log\frac{\Lambda^2}{m^2} - \frac{5}{4} - 2\int_0^1 dz\,(2-z) + 2\int_0^1 dz$$

$$-\int_0^1 d\log\left[(1-z)^2m^2+\mu^2\right]$$

$$\therefore \frac{2\pi}{\alpha}\,\delta Z_2 = -\frac{1}{2}\log\frac{\Lambda^2}{m^2} - \frac{5}{4} - 2\left(\frac{3}{2}\right) + 2 - \log\left(\frac{m^2+\cancel{\mu^2}}{\mu^2}\right)$$ ($\cancel{\mu^2}\to\phi$)

or $$\boxed{Z_2 = 1 - \frac{\alpha}{2\pi}\left(\frac{1}{2}\log\frac{\Lambda^2}{m^2} + \log\frac{m^2}{\mu^2} + \frac{9}{4}\right)}$$

⊛ Itzykson + Zuber Eq. (7-34)!

⊛ Notice that perturbative corrections to Z_2 make it < 1, although the I.R. logarithm technically a tiny bit worrisome (though not really: Z_2 disappears from all physical quantities)

⊛ This "trend" is sensible since:

$$\bar{\psi}^{\text{full}}(x)\,|0\rangle = \sqrt{Z_2}\,|\text{electron state}\rangle + \text{others},$$

so $Z_2 < 1$ "leaves room" for the other states

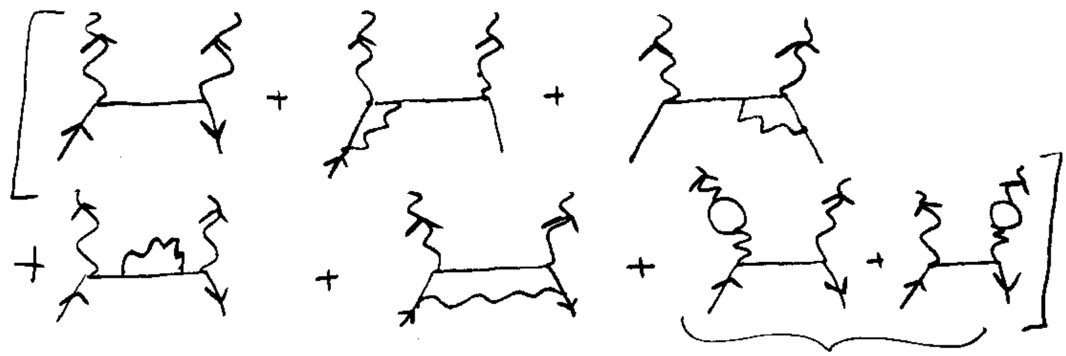
⊛ In reality, the "non perturbative" electron propagator structure is more complicated: the electron pole, and the $e+\gamma$ branch cut are not separated when $\mu\to 0$ (see Sect. 7.1, 7.2)

Finally : $\boxed{\text{Propagator } Z_2\text{'s \underline{also} absorbed}}$

⊛ We've seen how <u>external</u> line Z_2's are absorbed.
What about propagators on <u>internal</u> lines ?

⊛ Consider radiative corrections to $e^- e^+ \to \gamma\gamma$:

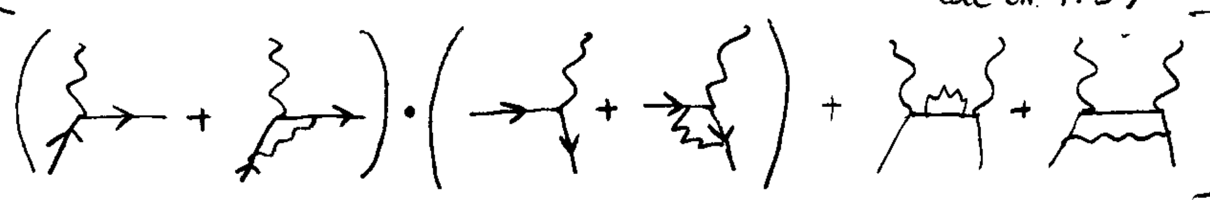
$$\mathcal{M}^{(1)} = \left(\sqrt{Z_2}\right)^2 \Big[\ldots \Big]$$



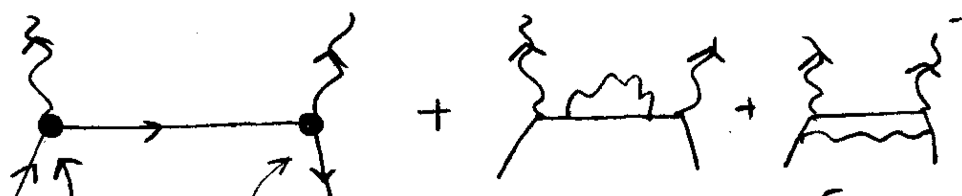
(due to external line renormalization)

⊛ forget these for now (see Ch. 7.5)

$$\approx Z_2 \Big[\ldots \Big]$$



$$= Z_2 \Big[\left(-ie_0 Z_1^{-1}\right)^2 \ldots \Big]$$



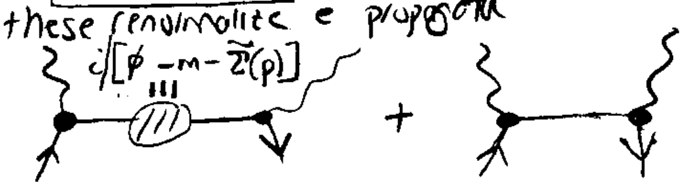
$\left(F_1^{\text{physical}}(q^2)\,\gamma^\mu + i F_2^{\text{physical}}\,\sigma^{\mu\nu} q_\nu / 2m\right)$ = U.V. finite

⊛ Now since the "effective vertex" • differs from the tree vertices here by $O(\alpha)$ (same for factors of Z_1) and since the extra graphs <u>already</u> of $O(\alpha^2)$:

$$\approx Z_2 \left(-ie_0 Z_1^{-1}\right)^2 \Big[\ldots \Big]$$

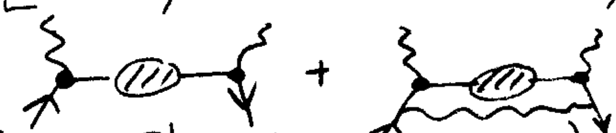
these renormalize e^- propagator

$$= Z_2 \left(-ie_0 Z_1^{-1}\right)^2 \Big[Z_2 \ldots + \ldots \Big]$$



$i/[\not p - m - \tilde{\Sigma}(p)]$

$$\approx \big(\underbrace{-ie_0 Z_2 Z_1^{-1}}\big)^2 \times \Big[\ldots \Big] = \text{U.V. finite!}$$



= e renormalized (actually $Z_2 Z_1^{-1} = 1$ by W/T identity)