Sets, Subsets, and the Empty Set: Students’ Constructions and Mathematical Conventions

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This study investigates students’ understanding of the basic concepts of introductory set theory: set, set element, cardinality, subset, and the empty set. The data was collected from a group of preservice elementary school teachers by means of written assessment, clinical interviews, and students’ participation in a computer-based project. The project included experimentation with basic set concepts in an open computer-based environment with mathematical computer language ISETL. A constructivist-oriented framework was used in analyzing the data. The results reveal complexities in students’ understanding, especially when set elements involved are sets themselves. Special attention is given to the description of students’ difficulties with the concept of the empty set.

The primary concern of this study is with learners’ understanding of mathematical concepts. Specifically, the learners are preservice elementary school teachers and the concepts are underlying sets. Within the growing body of literature on research in undergraduate mathematics education, including research on teacher education and concept formation, we found only one reference—Baxter (1994)—with an explicit focus on the concepts related to set theory. Baxter presented an “action research” with a goal for enhancing teaching and learning via improving the design of learner-centered materials. “Clearly,” stated the editors in their comments on Baxter’s paper, “it is a beginning for the process of building general and theoretically
coherent explanations of particular phenomena....Much work needs to be done, work that would build the theory framing the regularities and invariants” (Kaput & Dubinsky, 1994, p.97). Our study takes a step in this direction.

BACKGROUND

In the late ’60s, following the new math and the recommendations of various committees of the National Council of Teachers of Mathematics (NCTM) and the Mathematical Association of America, the principles of set theory became a part of the school mathematics curriculum (Feinstein, 1973). Some mathematicians treated this development as a “fashion injected” into the elementary school system (Heritage, 1974). In the literature from the ’70s we’ve found extensive discussion on how to teach sets meaningfully (Cable, 1985; Dienes, 1969; Feinstein, 1973; Freudenthal, 1969; Kapedia, 1976; Oberlin, 1970). We’ve also found reports on misinterpretation of the set concept by elementary school teachers and inappropriate uses of the set concept in elementary school textbooks (Damarin, 1977; Freudenthal, 1973; Pinker, 1981). Such evidence raises a question about the necessity and the value of teaching principles of set theory at the elementary school level.

In 1981, as a concluding remark to the article that investigated the assimilation of the concept “set” in the elementary school mathematics texts, Pinker wrote:

It is quite clear that the concept of set is not an easy one, and that it was not well assimilated [by preservice teachers] in spite of a protracted effort....Should it be introduced in the elementary school at all? (p. 462)

Should it? As a matter of fact, set theory has almost disappeared from elementary school mathematics textbooks in the late ’80s and early ’90s. Set theory concepts are not even mentioned in the 1989 NCTM Curriculum and Evaluation Standards for School Mathematics—one of the most quoted references in mathematics education lately. On the other hand, almost every mathematics course for preservice elementary school teachers includes a topic of introductory set theory. All the mathematics textbooks for preservice elementary school teachers that we examined (Gay, 1992; Krause, 1991; Musser & Burger, 1994—to list just a few of the recently published examples) included a chapter on set theory. The concepts of set theory are not taught to preservice teachers to enable them to teach these
concepts to their elementary school students. Why are they part of the curriculum? Should they stay as such?

We examined several arguments for and against the teaching of sets to preservice teachers. Most of the standard topics for preservice elementary school teachers, such as geometric shapes and transformations, rational numbers, introductory number theory, and data analysis are somehow related to the topics approached in elementary school. The concepts of set theory seem to be the only exception, the “leftovers” from the new math’s trend to introduce addition and subtraction via sets. It may appear that the curriculum for preservice elementary school teachers has not caught up with the curriculum for elementary school students. On the other hand, one may consider set theory concepts as an important indoctrination to the foundations of mathematics, one that extends mathematical horizons of preservice teachers beyond what they are expected to teach.

We do not intend to take a stand on this issue, but to respect the facts: Regardless of the reasons for which the introductory concepts of set theory are part of preservice elementary school teachers’ curriculum, many teachers in their mathematics education come across set theory concepts and acquire knowledge about them. The goal of this research is to investigate the nature of this knowledge and how it may be constructed by preservice elementary school teachers. Specifically, this study focuses on the concepts of sets, set elements, set cardinality, subsets, and the empty set. We hope that the findings of this research are relevant for teaching and learning of introductory discrete mathematics at all levels.

**WHAT ARE SETS?**

Most mathematicians do not attempt to define the term set treating the terms “set,” “element,” and “belonging” as undefined terms, according to Hamilton (cited in Feinstein, 1973). In a typical undergraduate level textbook on set theory, a set is defined as “a collection of things (called its members or elements), the collection being regarded as a single object” (Enderton, 1977, p.1). The undefined mathematical terms of set, element, and belonging rely on primitive experiential notions like “collection,” “object,” and “in.” Enderton’s cautionary note that the sets themselves are to be taken as a single object is an important clarification of how students are intended to treat these primitive notions. However, is this the case?
THEORETICAL FRAMEWORK

The particular interpretation of constructivism used in this study is based upon Dubinsky’s (1991) action-process-object developmental framework. Dubinsky developed this framework as an adaptation of some of Piaget’s (1965) central ideas to the studies of advanced mathematical thinking. The essence of this theoretical framework is that an individual, disequilibrated by a perceived problem situation into a particular context, will attempt to reequilibrate by assimilating the situation to existing schemas or, if necessary, reconstruct particular schemas enabling the individual to accommodate the situation. Dubinsky holds that the constructions are mainly of three kinds—actions, processes, and objects. An action is any repeatable physical or mental manipulation that transforms objects in some way. When the total action can take place entirely in the mind of an individual, or just be imagined as taking place without necessarily running through all of the specific steps, the action has been interiorized to become a process. New processes may be constructed by inverting or coordinating the existing processes. When it becomes possible for a process to be transformed by some action, then we say that it has been encapsulated to become an object.

Previously, this framework had been used in studies of undergraduate mathematics topics such as calculus and abstract algebra (Ayers, Davis, Dubinsky, & Lewin, 1988; Breidenbach, Dubinsky, Hawks, & Nichols, 1991; Dubinsky, Dautermann, Leron, & Zazkis, 1994). Also, a claim has been made that this framework is applicable for investigating the development of mathematical understanding in general, and should not be limited to the domains of “advanced” mathematical understanding. An attempt has been made to use this theoretical framework to describe the construction of knowledge of preservice elementary school teachers on topics related to the place value numeration system (Zazkis & Khoury, 1994) and introductory number theory (Zazkis & Campbell, 1994, 1996). The reader is advised to consult the above mentioned references for a more detailed description and exemplification of the action-process-object framework.

DATA COLLECTION

Participants in our research were preservice elementary school teachers enrolled in the professional development course Foundation of Mathematics for Teachers. The chapter on set theory, which included the basic
concepts of set, set cardinality, subset, Venn diagrams, union and intersection of sets, and the empty set, was a part of their core curriculum. The data was collected from three main sources: (a) written assessment, (b) ISETL project, and (c) clinical interviews.

**Written Assessment**

Shortly after the completion of the set theory topic, an assessment questionnaire was collected from 46 students. The questionnaire is given in Figure 1.

![Assessment questionnaire](image)

Students’ time to complete the questionnaires was not limited, and the majority completed the work within 15 minutes. The students were encouraged to “explain” their decisions, although some participants ignored this request. All students’ answers were recorded in terms of the True/False decision and its correctness according to the mathematical conventions. All students’ explanations and justifications were summarized.
Students enrolled in the Foundations of Mathematics for Teachers course had to choose one of two projects as a part of their course work. About one half of the students chose the “ISETL project”—a project related to the set concepts, based on computer experience with the computer language ISETL. In the next paragraph we introduce ISETL and then provide a description of the project.

ISETL is a powerful interactive computer language that was created as a tool for teaching and learning mathematics. Its syntax is very close to the syntax of standard mathematical notation, and the claim has been made that interactive computer experience with ISETL helps students in constructing mathematical concepts (Ayers et al., 1988; Baxter, 1994). Programming in ISETL has been an integral part of several courses for mathematics and science majors in discrete mathematics, calculus, and abstract algebra, and the results of such integration are encouraging (Breidenbach et al., 1991; Leron & Dubinsky, 1995). Our study is a pilot attempt to integrate ISETL experience into a mathematics course for preservice elementary school teachers. We do not wish to evaluate this experience and conduct a formal comparison between the achievements of students who have chosen ISETL project and those who have not. Rather, we use this experience as an additional tool to investigate students’ construction of the underlying concepts of set theory. The ISETL code for sets, set formations, set elements, cardinality, subsets, and union and intersection of sets is summarized in Figure 2.

The students were provided with worksheets to learn the ISETL code using a “hands-on” approach. Students were asked to read the code, try to predict the computer output, check their prediction, and explain the result to their partner. In addition, students were provided with a variety of exercises to implement and enhance their learning. The first 2 hours of their computer experience the students spent in the computer lab with their instructor; afterwards they had unlimited access to the workshop where computers and help were available. The students worked in groups of three or four; they were encouraged to experiment and to check their conjectures with the help of a computer. Following the introduction to the language, the students were invited to work on several problems using the ISETL code. A list of problems is provided in Appendix A. The problems varied from assignment exercises, for example,

There are at least three different ways to define in ISETL the set of numbers
\{5,10,15,20,25,30,35,40,45,50,55,60,65,70,75,80,85,90,95,100\}

What are they? Verify with ISETL that the sets you’ve defined are indeed equal.
to more open discussion, like

Consider the empty set: {}.

Write a paragraph about the empty set and its properties. Demonstrate these properties with examples using the ISETL code.

<table>
<thead>
<tr>
<th>Defining sets:</th>
<th>Cardinality of Sets</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ A := {1, 2, 3, 4, 5}; ]</td>
<td>[ #A; ]</td>
</tr>
<tr>
<td>{1, 2, 3, 4, 5};</td>
<td>5;</td>
</tr>
<tr>
<td>[ B := {2, 3}, {4, 5}];</td>
<td>[ #B; ]</td>
</tr>
<tr>
<td>{2, 3}, {4, 5};</td>
<td>3;</td>
</tr>
<tr>
<td>[ Y := {2, 3, 2, 3, 2}; ]</td>
<td>[ #Y; ]</td>
</tr>
<tr>
<td>{2, 3};</td>
<td>2;</td>
</tr>
<tr>
<td>[ Z := {}; ]</td>
<td>[ #Z; ]</td>
</tr>
<tr>
<td>{};</td>
<td>0;</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Elements of sets:</th>
<th>Subsets:</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ 3 \text{ in } A; ]</td>
<td>[ {4, 5} \text{ subset } A; ]</td>
</tr>
<tr>
<td>true;</td>
<td>true;</td>
</tr>
<tr>
<td>[ {3} \text{ in } A; ]</td>
<td>[ {4, 5} \text{ subset } B; ]</td>
</tr>
<tr>
<td>false;</td>
<td>false;</td>
</tr>
<tr>
<td>[ {3} \text{ in } B; ]</td>
<td>[ Z \text{ subset } B; ]</td>
</tr>
<tr>
<td>true;</td>
<td>true;</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Union and Intersection:</th>
<th>Set Formation:</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ A \cup B; ]</td>
<td>[ K := {i</td>
</tr>
<tr>
<td>{1, 2, 3, 4, 5}, {3}, {4, 5};</td>
<td>{2, 4};</td>
</tr>
<tr>
<td>[ A \cap B; ]</td>
<td>[ S := {i^2</td>
</tr>
<tr>
<td>{2};</td>
<td>{1, 4, 9, 16, 25};</td>
</tr>
</tbody>
</table>

Figure 2. Set concepts in ISETL

All the problems were solvable using the direct mode code, not programming. The reason for this was twofold: first, we felt that a great deal of mathematics can be done in ISETL using the direct mode code; second, due to limited time and lack of programming experience of this population of students, we attempted to focus on mathematical ideas by avoiding programming at this stage. Again, the purpose of this study is NOT to examine students’ success with the specific problems, rather it is to illustrate students’ understanding of set concepts by using some of their responses to specific questions.
Interviews

Semi-structured interviews were conducted with 15 out of 28 students who undertook the ISETL project. The interviews were informal and about 15-20 minutes in duration. In the interviews the students were asked to present specific problems they worked on in their project, to reflect on their “ISETL experience,” and also to discuss some general set concepts, such as

- What is a subset?
- How would you check if one set is a subset of another?
- What does “empty set” mean to you?
- Why is the empty set a subset of any set?

Most of the interviews were audiotaped and transcribed. Three interviews, following participants’ request, were not audiotaped, but summarized by the interviewer immediately after the interview. During the interviews the students were prompted for understanding that may not have been apparent from their initial responses. They were also encouraged to use the computer to verify their conjectures or to exemplify their claims.

RESULTS AND INTERPRETATIONS

Our analysis focused on the following issues: (a) set elements, (b) set cardinality, (c) subsets, and (d) the empty set. Each one of these issues is presented in a subsection. In each section the results are drawn from the relevant parts of all three data sources.

Sets and Set Elements

The participants had no difficulty in identifying elements of “flat” sets, that is, sets that have no sets as elements, such as $S = \{1, 2, 17, 69\}$. However, more convoluted sets revealed deficiencies in students’ understanding of the concept of the set element. Table 1 summarizes students’ responses on items related to identifying the elements of the Set A.
Table 1
Summary of Responses on Items 1-5: Identification of Set Elements

<table>
<thead>
<tr>
<th>Question</th>
<th>Truth Value</th>
<th>Correct responses #</th>
<th>Correct responses %</th>
<th>Typical arguments</th>
</tr>
</thead>
<tbody>
<tr>
<td>A= {5,7, {5}, {5, 7, {7} } }</td>
<td>T</td>
<td>46</td>
<td>100%</td>
<td>(T) 5 is part of the set (T) It is one of the numbers included within Set A (T) It is true because you can find it alone in the set (T) Yes, because 5 is in the main set of brackets of Set A (T) Yes, because it is within the outer brackets with no brackets around it</td>
</tr>
<tr>
<td>2. 7 ∈ A</td>
<td>T</td>
<td>45</td>
<td>98%</td>
<td>responses similar to Item 1</td>
</tr>
<tr>
<td>3. {5} ∈ A</td>
<td>T</td>
<td>39</td>
<td>85%</td>
<td>(T) {5} is part of the set (T) Yes because {5} is in the main set of brackets (F) {5} is a subset (F) It is a set within a set, not an element</td>
</tr>
<tr>
<td>4. {7} ∈ A</td>
<td>F</td>
<td>28</td>
<td>60%</td>
<td>(F) {7} is an element in set {5,7, {7}} (F) It is an element within a subset of A (F) It is a subset of an element (T) The element {7} is found in Set A</td>
</tr>
<tr>
<td>5. {5, 7, {7}} ∈ A</td>
<td>T</td>
<td>40</td>
<td>87%</td>
<td>(T) Subset existing as an element (F) It is a subset, not an element</td>
</tr>
</tbody>
</table>

There was a mutual agreement among the participants that 5 and 7 were indeed the elements of the set A={5, 7, {5}, {5, 7, {7} } }. The majority of participants based their explanations on what is found or included in Set A. Also, more visual explanations that referred to the main or outer
brackets of A were popular. We see that the action construction of a set for these students consisted of listing the elements between the brackets.

With \{5\} and \{7\} and \{5,7,\{7\}\}, the decisions were more complex. There was an evident similarity between students’ responses to Items 3 and 5, that is, in deciding whether \{5\} and \{5,7,\{7\}\} were elements of A. Every student except one, who made the wrong decision on Item 3 followed up with the wrong decision on Item 5. The argument that supported this decision was “It is a subset, not an element.” It seems like a number of students were still struggling with the idea that sets can have other sets as elements. We believe that for these students the idea of a set was not perceived as an object that could appear as an element of another set.

The statement of \{7\} being the element of A seemed to be the most problematic, as it brought only 28 correct responses. In fact, \{7\} is not an element of A, but it is an element of \{5,7,\{7\}\}, which is one of A’s elements. Since many students thought of elements of a set as something that is “found in” the set, the symbol \{7\} is indeed there. Therefore, it is not surprising that those individuals incorrectly identified \{7\} as an element. It appears that these students were not able to treat a set—in this case the set \{5,7,\{7\}\}—as one object without paying attention to its components. This indicates that for these students a set is an action or a process, and the object of a set has yet to be encapsulated.

It may appear that repeated appearances of 5s and 7s in the Set A may have added more complexity to the set structure. However, from our informal discussions with students and in-class observations, we presume that similar results would be obtained had the set been \{5,7,\{6\}, \{8,9,\{2\}\}\}.

We find additional evidence for action construction of a set in the following excerpt from the interview with Nicole. Nicole claimed that \{6\} was an element of the set \{6, \{6, \{6\}\} \}, “seeing” it in the third listed 6.

Interviewer: Is \{6\} an element of \{6, \{6, \{6\}\} \}?
Nicole: Yes.
Interviewer: How do you know?
Nicole: It’s an element on its own. It has a comma before it and a bracket after it.
Interviewer: Where do you see it?
Nicole: It’s the third 6.

Returning to the Set A, more than half of the justifications for the correct decisions mentioned that even though \{7\} was not an element of A, it was an element of an element \{5,7,\{7\}\}. Mathematically, this remark, though correct, is irrelevant, but psychologically it seems to have a high
explanatory power. Also, such a statement seems to be an indication of the misconceptions that students might have held in the past and managed to overcome. Similar justification we find in the interviews with Andy and Susan, who pointed out that \( \{6, \{6\}\} \), but not \( \{6\} \), was an element of \( \{6, \{6, \{6\}\}\}\) :

Interviewer: Look at the set, \( \{6, \{6\}\} \) is an element but not \( \{6\} \)....Is \( \{6\} \) an element?
Andy: No.
Interviewer: Why not?
Andy: Because it looks like it is here (pointing to the third 6) but really this is one whole set so that it is not an element. If it was on its own, it would be, but it's not.

Interviewer: Is \( \{6\} \) an element of \( \{6, \{6, \{6\}\}\}\)?
Susan: No.
Interviewer: Why not?
Susan: \( \{6, \{6\}\} \) is an element but not \( \{6\} \).

When one tries to decide whether \( \{7\} \) is an element of the Set A using Andy’s observation, indeed “it looks like it is there.” We would like to offer an explanation as to what may have caused the belief that \( \{7\} \) was an element of A—a reference to which we find in both correct and incorrect answers. We believe that additional difficulty in identifying \( \{7\} \) as not being an element of A is caused by the tendency to consider the statements “is an element of” or “belongs to” as transitive relations. Such a tendency is influenced by a student’s experience in real life situations: If Cantor is a player in a team of mathematicians, which is a team in the college league, he is considered to be a player in a college league; if I have an aspirin in my cosmetics bag which is in my purse, I would claim to have an aspirin in my purse; if we live in Vancouver, which is in Canada, we definitely live in Canada. In a counter analogy, if there is an element \( \{7\} \) in a set \( \{5,7,\{7\}\}\), which is an element of a Set A, \( \{7\} \) is NOT considered by mathematical convention to be an element of A. Following the out-of-set-theory experiences, the transitivity is often assumed by the learner and has to be rejected by reconstructing nontransitivity.

Cardinality

In early childhood acquisition of mathematical concepts, classification skills are considered to be precounting skills and serve as a prerequisite for
counting skills: A child should be able to decide which objects belong to the set of the objects being counted in order to count them. Analogously, classification seems to be a prerequisite for establishing cardinality of a given set, that is, for counting its objects. Classification here is the ability to create a dichotomy between elements and non-elements of a given set. The problem is to recognize as set elements objects of a different type. Consider for example the following set of drinks: \( D = \{ \text{Coca-cola can, 6-pack of beer} \} \). This raises the question of set cardinality, that is, how many elements are there in \( D \). Are there two elements in \( D \) or seven? The set theory answer is two, even though seven may seem in this example more close to the learner’s intuition and experiences.

There are two concerns in specifying set cardinality: first, what constitutes an element and second, how to treat repeating objects. The latter was one of the main issues in the criticism of the teaching of set theory and using sets like \( \{***\} \) for models of addition and multiplication (Feinstein, 1973; Pinker, 1981). By the idea of using \( \{***\} \cup \{**\} \) as a model for 3+2 at the same time when \( \{1,1,1\} = \{1\} \), the inconsistency is suggested, and the requirement for disjoint sets is ignored. As a consequence, a suggestion to drop the reference to sets in elementary school was made. The first concern of what constitutes an element—exemplified by whether a 6-pack of beer counts once or six times—is one of the issues discussed in this research.

When asked about the number of elements of Set A, students responses varied from two to eight. As shown in Table 2, 7 out of 46 students didn’t answer this question, and 23 students gave the correct answer, claiming the number of elements of A was four.

**Table 2**

Summary of Students’ Responses to the Number of Elements of the Set A

<table>
<thead>
<tr>
<th>n(A)</th>
<th>Frequency of response</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>23</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
</tr>
<tr>
<td>No answer</td>
<td>7</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>46</strong></td>
</tr>
</tbody>
</table>
The claim \( n(A) = 2 \) was made by 6 students who most likely identified only 5 and 7 as elements of \( A \). One student explicitly mentioned that multiple listing of the set elements does not increase the number of elements. The answer \( n(A) = 3 \) may show that no distinction was made between a number 5 and single element set \{5\}. The answer \( n(A) = 6 \), repeated by 3 students, is probably derived by counting number symbols appearing in Set A, without recognizing the set brackets. Students who claimed \( n(A) = 5 \) identified \{7\} among the elements of \( A \). This is consistent with our conjecture of assumed transitivity, discussed in the previous section. We believe that the single answer \( n(A) = 8 \) was a miscalculation.

Also we would like to mention here that some of the students’ responses were not entirely consistent. For example, there were students claiming that \( A \) had two elements, but, in circling True for the first four parts of the questionnaire, they identified at least four different elements of \( A \). Again, this may be taken as an evidence that the distinction between a number and a single element set was not made.

**Subsets**

Students’ responses to the questions related to the subsets of \( A \) are summarized in Table 3. Items 13 and 14 may appear misleading, however, they uncover students’ thinking about sets which may not be apparent with “straight forward” examples.

A closer look at a very high rate of success—96%—with Question 7 reveals that in about one third of the cases, the right choice was made for the wrong reason. While \{5\} is obviously a subset of \( A \) since 5 is one of the elements, some students claimed that \{5\} was a subset by looking at the element \{5\}. This explains the discrepancy between the decisions made on Statement 7—\{5\} a subset of \( A \)—and the decisions on Item 8—\{7\} a subset of \( A \): Students who made the correct decision for the wrong reason on Item 7, made a wrong decision on Item 8. Again, looking at \{7\} and not at 7, the claim was that \{7\} was an element within a subset \{5, 7, \{7\}\}, not a subset on its own. Other responses, like “it is a subset of a subset” indicated apparent confusion between a concept of an element and a concept of a subset.
### Table 3
Summary of Responses on Items 7, 8, 11-14: Identification of Subsets

<table>
<thead>
<tr>
<th>Statement</th>
<th>Truth Value</th>
<th>Correct responses # (n=46)</th>
<th>Correct responses %</th>
<th>Typical arguments</th>
</tr>
</thead>
<tbody>
<tr>
<td>7. {5} \subseteq A</td>
<td>T</td>
<td>44</td>
<td>96%</td>
<td>(T) 5 is an element, therefore {5} can be a subset (T) For the first 5 listed in A (T) {5} is in a set A so it is a subset</td>
</tr>
<tr>
<td>8. {7} \subseteq A</td>
<td>T</td>
<td>29</td>
<td>63%</td>
<td>(T) 7 is an element so {7} is a subset (F) {7} is an element within a subset {5,7,{7}}, not a subset on its own (F) This is subset of a subset</td>
</tr>
<tr>
<td>11. {(5)} \subseteq A</td>
<td>T</td>
<td>25</td>
<td>54%</td>
<td>(T) Yes, because {5} is an element of A</td>
</tr>
<tr>
<td>12. {(7)} \subseteq A</td>
<td>F</td>
<td>30</td>
<td>65%</td>
<td>(F) Because {7} is not an element of A (T) {7} is inside 2 sets {}\</td>
</tr>
<tr>
<td>13. {5, 7, {7}} \subseteq A</td>
<td>F</td>
<td>7</td>
<td>15%</td>
<td>(F) { {5,7,{7}} } would be a subset (F) It is an element. It would need extra brackets to be a subset (T) This is one element in A (T) This is a set within a set therefore it is a subset</td>
</tr>
<tr>
<td>14. {5, 7, {5}} \subseteq A</td>
<td>T</td>
<td>19</td>
<td>41%</td>
<td>(T) They are all elements in A (T) The first 3 elements will make this subset (F) The brackets are not around these three elements in particular (F) It cannot be found in Set A</td>
</tr>
</tbody>
</table>
For an example of a correct answer wrongly justified, we refer to the interview with Susan. She claims that \{6\} was a subset of \{ 6, \{6, \{6\} \} \}, but points to the “third 6” to justify her decision.

Interviewer: What is a subset?
Interviewer: Can you give me an example?
Susan: Set A consists of 1,2,3,4,5 and Set B of 2,3. B is a subset of A.
Interviewer: Is \{6\}—let me write it down here—a subset of \{ 6, \{6, \{6\} \} \}?
Susan: Yah.
Interviewer: Why?
Susan: Because that is contained within the set.
Interviewer: You’re pointing to the third 6.
Susan: Yes.

Actually, for about one third of participants, the concept of a subset was confused with an element of a set which is itself a set. This confusion caused 85% of our participants to claim that \{5, 7, \{7\}\} was a subset of A, when actually this is one of the elements. In the interview with Sandra this confusion is explicit:

Interviewer: What is a subset?
Sandra: A subset is an element inside of the set that contains its own elements.
Interviewer: Can you please give an example?
Sandra: For example: A= \{1,2,3,\{4,5\}\}. \{4,5\} is a subset of A.
Interviewer: Aha, would you say that \{1,2,3\} is subset of A?
Sandra: No. I believe no.

We find further evidence to this confusion in other interviews. In the following example, Stanley gives a reasonable definition of a subset, but claims that \{6\} was not a subset of \{6, \{6,\{6\}\}\}. Similarly to Susan, this decision is made considering the symbol \{6\} and not 6.

Interviewer: What is a subset?
Stanley: It is any number of elements contained within a set including the empty set.
Interviewer: Can you give me an example?
Stanley: Let Set A be \{1,2,3,4,5\}. Let B be \{2,3,4\}. B is a subset of A.
Stanley’s justification that “\{6\} would have to be a subset of \{6,\{6\}\}” is also false. It seems that Stanley, like Sandra, includes an element which is itself a set among subsets, even though this is not done exclusively and is not claimed explicitly.

Nicole seems to grasp the idea of a subset, even though her explanation of what is a subset excludes the set as a subset of itself. Nicole claims that \{6\} was a subset of \{6, \{6, \{6\}\}\} “because of the first 6,” that is listed as an element is the set.

Even though Nicole’s words “set within a set” are ambiguous, we believe her further explanation and her answer demonstrate understanding of the subset idea and ability to use a powerful heuristic strategy to justify her ideas. Also, her reference to “learning from the computer” is encouraging.

Allison was probably the only interviewee that, when asked what a subset was, replied with a statement close to the conventional definition.
However, the majority of students, as we’ve seen several examples through this section, described subsets in a way meaningful for them, which at times differed from the conventional one not only in wording, but also in meaning. Andrea demonstrated a typical action construction for a subset. It seems that for her, a subset of \{1,2,3\} is being created by writing the symbol \{1\} with brackets around it.

**Interviewer:** What is a subset?

Andrea:
I’ll give you an example. If you had a set that had the elements 1,2,3, a subset would be one of those elements so if you were to write 1 with brackets around it, that would be a subset.

**Interviewer:** How do you check if something is a subset?

Andrea:
I would look at the set and each individual element could be a subset. Also a subset could be a combination, like 1,2 could be a subset of the thing I mentioned but the whole set isn’t a subset.

**Interviewer:** Look at the set, \{6, \{6, \{6\}\}\}. Is \{6\} a subset?

Andrea:
I don’t think it is... just because in order to be a subset it would have to be on its own but it’s not.

**Interviewer:** What about the first six?

Andrea:
It’s a subset and an element, when you take it out individually you can make it to a subset.

Further in her interview the action is intended and probably the process of a subset is being interiorized. For Andrea, \{6\} is not yet a subset of \{6, \{6, \{6\}\}\} but one can “make it to a subset.” Acquiring an understanding that there are subsets for every set, regardless of whether they are “made” or not, would indicate a step toward encapsulation of a subset to an object. This is probably a further challenge for Andrea and some of her classmates.

**Empty Set**

The long way “From One to Zero” (Ifrah, 1985) is acknowledged in the history of numbers. In a counter-analogy, the concept of an empty set was an integral part of the development of set theory. However, our data suggests that for many students there is a “long way from a set to the empty set,” that is, many students who have a reasonable understanding of sets and subsets experience difficulties in constructing the concept of the empty set.
Table 4
Summary of Responses on Items 6, 9, and 10: The Empty Set

<table>
<thead>
<tr>
<th>Question</th>
<th>Truth Value</th>
<th>Correct responses # (n=46)</th>
<th>Correct responses %</th>
<th>Typical arguments</th>
</tr>
</thead>
<tbody>
<tr>
<td>6. $f \in A$</td>
<td>F</td>
<td>42</td>
<td>91%</td>
<td>(F) There is no empty set shown in the set, therefore it can't be an element (F) The empty set is not mentioned anywhere in the set (F) Not found in A so it's not an element</td>
</tr>
<tr>
<td>9. $f \subseteq A$</td>
<td>T</td>
<td>20</td>
<td>43%</td>
<td>(T) Empty set is a subset of any set (T) Empty set is a subset of all sets (T) Empty set is always considered a subset of any set (F) There is no empty set indicated in A (F) There is no empty set indicated in the question</td>
</tr>
<tr>
<td>10. ${f} \subseteq A$</td>
<td>F</td>
<td>38</td>
<td>83%</td>
<td>(T) Empty set is a subset of all sets (F) ${f}$ is not an empty set (F) $f$ is not an element in A (F) There is no empty set within A</td>
</tr>
</tbody>
</table>

Of the three items on the written assessment involving the empty set (Items 6, 9, and 10), the majority of students (42 out of 46) gave a correct answer on Item 6, claiming False to the statement “$f$ is an element in Set A” (see Table 4). Students who circled True chose not to explain their decision. Phrases like “not found,” “not shown,” and “not mentioned” as typical explanations for the choice of False point to the student’s action construction of sets, that is, putting together the elements and a tendency to see all the elements explicitly listed. Several remarks explicitly indicated students’ confusion: “I really don’t understand if these are in the set or not.” Others demonstrated confusion, without acknowledging it explicitly: “Empty set cannot be an element, because the empty set is a set of no elements.”
On Item 9, 20 students out of 46 answered correctly, claiming True to the statement “f is a subset of Set A.” Most of the True claims referred to the rule students were exposed to. We will show further in this section, that while these students indeed learned the rule, they may not quite understand why mathematicians support such a consideration.

The typical justifications for the False claims on this item indicated that students expected to see the empty set listed in A in order to respond True. These claims may be additional indication of students’ confusion between the notion of an element in a set and a notion of a subset that is ambiguous in the use of the words “in A.”

On Item 10, 38 out of 46 students responded correctly with False to the statement “{f} is a subset of Set A.” At first, we found the significantly bigger rate of success on the 10th item in comparison to the 9th item (43% vs. 83%) surprising. On the other hand it is in line with the high (91%) rate of success on Item 6. After examining students’ written comments we would like to offer some explanations to this phenomenon. Students’ justifications for the incorrect choice True indicated their confusion between \{f\} and f, that is a set with one—the empty set—element and the empty set itself. Mostly, they repeated the argument for the previous item, claiming that the empty set was a subset of all sets. The arguments for the choice of False varied. We learned that the correct choice of the answer was sometimes caused by the wrong reasoning. Some students looked for the f to be listed in A, and since it was not, drew their conclusion. Actually, 17 out of 46 students replied False to all the three items #6, 9, and 10, justifying with “there is no empty set within A.” Other responses described possible construction of an action for a subset: You choose an element(s) from the set, wrap it in curly brackets, and get a subset. Consequently, if f is not an element of A, then \{f\} is not a subset of A.

What is an empty set? By definition it is a set that has no elements. When students were asked to list properties of an empty set, the first one that was usually mentioned was that the empty set was a subset of every set. However, this property was worded by some students as “empty set is in every set”; that probably was caused by the confusion between a subset and an element in a set. In the excerpt below, Andrea claims that the empty set is “in every set,” but the computer (ISETL) doesn’t know that.

Interviewer: What is an empty set?
Andrea: Set with no elements.
Interviewer: And what do you know about the empty set?
Andrea: It’s in every set.
Interviewer: And what do you mean by that?
Andrea: It is like in class we said, and in the book, we were told that an empty set is in every set, but when we asked the computer, it said no.

Interviewer: And how did you ask the computer?

Andrea made the following conversation with ISETL:

```plaintext
> A := {1,2,3,4,5};
> A;
{1, 2, 3, 4, 5};
> {} in A;
false;
> 1 in A;
true;
```

Andrea: You see, the computer doesn’t know it unless it is listed as an element. It has to see the empty set explicitly in the set.

Interviewer: What do you mean by that?

Andrea demonstrated by modifying the Set A:

```plaintext
A := {1,2,3,4,{});
> {} in A;
true;
```

Andrea: You see, it has to be told explicitly.

Interviewer: And how do you explain this?

Andrea: Isn’t empty set a part of every set? It should be there, I remember this for sure.

Interviewer: So, how do you explain this conflict between your knowledge and the computer’s?

Andrea: It is not the computer’s fault, it is the person who programmed it. Computers have limitations, you know. So I just have to put it there for it. Not a big deal.

To check whether the empty set was indeed in every set, Andrea wrote `{}` in A, which is the ISETL code for making a statement $\emptyset$ is an element of the Set $A$. The computers output of False was interpreted as a lack of the computer’s knowledge and programmer’s fault. Later, following the interviewer’s explicit tutorial intervention, Andrea was able to make a distinction between empty set as an element and empty set as a subset. Actually, Andrea’s confusion with the empty set exemplifies the general confusion between the element in a set and a subset. We find a reference in the students’ ISETL projects that the ISETL experience helped them to clarify the confusion between the subset and an element, which is exemplified in the following citation from a student project: “An empty set is subset of every set whether or not it is an element within that set. Empty set is not an element of all sets unless it is presented in the set with brackets, `{}`.”
Several participants seem to unify the concepts of empty set and zero. In some cases the words “zero” and “empty set” were used interchangeably. One group of students wrote in their project, “Just as we can say that we have no cats in our house, we can say that we have no elements in our set. Zero is not an element of every set, while zero is a subset of every set.”

In the interview, a representative of this group explained that the group initially identified the symbol \( \in \) with the similarly looking symbol \( \notin \), specifying that the zero is crossed to distinguish between the number zero and the letter O. Another group made an attempt to find meaning in their confusion with symbols, explaining that “the empty set is written as \{\} or as 0 (zero), to show that the empty set has no elements.” In the following excerpt, it seems that Sandra’s confusion is more than confusion with symbols:

Interviewer: What is an empty set?
Sandra: An empty set is a set that has no elements. It is like nothing is there. Just empty brackets or \{\}.

Interviewer: And why do you say that \{0\} is an empty set?
Sandra: Oh, because zero is nothing, and it means nothing is there.

Interviewer: And how would you feel if I suggested that the set \{0\} has one element in it and it is the element 0 (zero)?
Sandra: But 0 is nothing so \{0\} is still empty.

Probably the source of Sandra’s confusion is in considering zero as “nothing.” In her case the empty set, that is, “set with zero elements,” was constructed as the “set with zero.” Candice explicitly acknowledges her confusion between empty set and zero, when asked to reflect on her experience.

Interviewer: What do you remember being difficult?
Candice: The empty set. The empty set and the plain zero. I didn’t understand the difference between the little zero with the thing in it and the zero in brackets. Or the brackets by themselves. I didn’t understand that just a plain zero is like an element—or it contains an element or it is an element in a set. Especially when there was just a zero inside the set. I didn’t understand what that was...you know how if there is a zero in the set they ask you to name the elements....it confused me.
Actually, when saying “just a zero inside the set,” Candice pointed on the \{\emptyset\} symbol. Probably her main difficulty was in distinguishing between \{\emptyset\} and \emptyset. We believe that the use of different symbols for the empty set—\emptyset in the class and \{\} in ISETL—contributed to her confusion.

**Why is the empty set a subset of every set?** Although it was stated correctly, but probably by rote, by the majority of the interviewees that the empty set was a subset of every set, no one provided a complete explanation as to why this was the case. Most of the students, like in the excerpts below, referred to the fact as a “rule of a thumb,” something they learned in class, something they were told was true, or something they learned from the ISETL experience.

Interviewer: Why is the empty set a subset of every set?
Nicole: Why it’s always there? I didn’t know. I just knew it was. It’s just given that empty set will be in every set.
Allison: I don’t understand that at all. I asked one of the TAs in the Math Lab, and I was told “it is.”
Candice: It’s something that I learned or it can be found by doing the True/False questions with ISETL.

We suggest that it is very hard to apply the definition of subset to the empty set. The difficulty comes from an “if-then” statement which is vacuous in the “if” part. In what follows we try to highlight the source of difficulty and suggest how pedagogical treatment may help to overcome it.

B is said to be a subset of A if every element of B is an element of A. More formal mathematical definition can be presented as an if-then statement: B is a subset of A if \(x \in B\) then \(x \in A\), for all \(x \in B\). This wording not only practices mathematical formalism, but also provides an algorithm, an action, for determining whether B is a subset of A:

1. Find any element \(x \in B\);
2. check whether \(x \in A\);
3. if it is, repeat Steps 1 and 2 for all other elements of B and conclude that B is a subset of A;
4. if it isn’t, conclude B isn’t a subset of A.

When B is an empty set, one cannot choose an element \(x\), such that \(x \in B\). That is, for any choice of an element \(x\), the statement \(P—x \in B—is false. Then, regardless of the truth value of the statement \(Q—x \in A—the
value of implication “if P then Q” is true. The conceptual and maybe even emotional difficulty with accepting the counterintuitive truth value True of implication “False ==> True” is known to the mathematics instructors. Actually, Dubinsky, Elterman, and Gong (1989) refer to this problem and claim that computer feedback from ISETL seemed to be more convincing for students in discrete mathematics course than any “logical” explanation.

The process to decide whether B is a subset of A can be exemplified using a computer code. In ISETL the code $A \subseteq B$ will generate the output of either False or True. But what if there were no “subset” primitive? One could define a procedure checking for a subset as follows:

```plaintext
Subset1:=func(A,B);
    return forall x in B|(x in B impl x in A);
end;
```

This procedure will output True if B is a subset of A and False otherwise. It uses “forall” control statement, as well as “impl” for implication, which are primitives in ISETL, but don’t exist in Pascal, for example. So, how can one define the subset procedure without using “forall” or “impl”? One possibility is given below:

```plaintext
Subset2:=func(A,B);
    x:=findelement (B);  
    if is_defined (x) then
        if x in A then return Subset2 (A, B less x);
        else return “false”;
    end;
    else return “true”;
end;
end;
```

```plaintext
findelement:= func (B);
    return arb(B);
end;
```

This recursive function is based on the explicit algorithm for checking for a subset described above. The recursive call defines a new Set B, which is the old Set B without the element x. In doing this, the cardinality of B is reduced each time Subset2 is activated. If there exists an element in B which is not in A, the procedure will return False. Otherwise, by reducing the cardinality of B in each recursive call, the local value of B will become the empty set. In this case there is no element x to be found in B or, in
ISETL terms, the “found element” \(x\) in not defined. Therefore the procedure will return True. Similarly, True is returned when \(B\) is an empty set in the beginning.

When \(B\) is an empty set, students don’t see an explicit process to be carried out. Referring to the computer program analogy from the previous paragraph—the procedure stops without going through the inner if-then loop even once. Inability to carry out some process may be additional reason for the recurring (26 out of 46) claim False to the statement “\(\forall x \in A\)” is a subset of Set A.” In case of subset definition, the statement “\(x \in B\)” implies that there is something to check whether or not it is in \(B\). The computer’s inability to “find element \(x \in B\)” leads to a simple solution of leaving \(x\) undefined. The student’s inability to “find element \(x \in B\)” may cause disequilibration. Students’ tendency to consider the empty set itself to be the \(x\), the object whose “in-ness” is checked, can be considered as one of the means to achieve equilibrium. The following example demonstrates this tendency.

Interviewer: Tell me about the empty set.
Allison: It can be found in every set...because there is some nothings in there. Also and 0 is kind of a nothing. There will always be nothing in there. Also the empty set is a set with no elements in it so the empty set is a set into itself.

[...]
Interviewer: In mathematics we say that a Set \(B\) is a subset of a Set \(A\) if every element of Set \(B\) is an element of Set \(A\). Could you use the definition to explain why the empty set is a subset of every set?
Allison: My instinct tells me I could...
Interviewer: Please try. [Interviewer writes down the definition of subset].
Allison: I see that. If the empty set is an element of every set it would have to be in Set \(B\). If every thing was in \(B\) then it would have to be in \(A\) as well.

When the Set \(B\) has no elements, the student has no objects to apply the subset definition to and therefore no process to be carried out. Essentially, Allison invents such a process by turning what should be a “\(F \implies T\)” argument, which is logically-mathematically true into a “\(T \implies T\)” argument, which is obviously true.
The following example from the interview with Susan seems to take a different perspective on the idea of the subset.

**Interviewer:** What can you tell me about the empty set?
**Susan:** It’s a subset of every set. It has no elements in it.
**Interviewer:** Why is the empty set a subset of every set?
**Susan:** I don’t know. I don’t know how to phrase it.
**Interviewer:** Take a stab at it.
**Susan:** Umh...The number of elements is 0 so it’s like having a subset of 0 elements.
**Interviewer:** How do you check if something is a subset?
**Susan:** You see if they are some of the elements of A.
**Interviewer:** Are all the elements of the empty set in A?
**Susan:** Yes, because there are no elements.
**Interviewer:** So it’s kind of by default.
**Susan:** Yah.

It seems that the “trivial application” of subset definition to the empty set is nontrivial mathematical activity. Actually, since our participants didn’t turn to definitions when checking for non-empty subsets, it is not surprising that they prefer to leave the definition aside when explaining why the empty set is a subset of any set A. One possible action construction for a subset, implicit in Susan’s words “You see if they are some of the elements of A,” is by selecting any number of elements from the original set. Analogously, the action of checking for a subset could be to determine whether a specific subset “can be selected” from any number of elements of the original set. This “any number of elements” varies at first from 1 to the cardinality of A. It is later understood that zero elements could be chosen to create the empty subset. The following excerpt exemplifies this kind of growth. (Leslie and Anna were interviewed in the same session).

**Interviewer:** What is a subset?
**Leslie:** Any portion of your original set. Minimum of 1 element and a maximum of all your elements.

[...]
**Interviewer:** Why is the empty set a subset of every set?
**Anna:** Because a subset is contained within a set and a set can have no elements.
**Leslie:** My former definition of subset is wrong. It can choose a minimum of zero, that is why the empty set is a subset of all sets.
Leslie says initially that a subset is a set with “minimum of 1 element and a maximum of all your elements,” that is, elements of the original set. Later, in order to justify the empty set being a subset of every set, Leslie changes her description of a subset to include the possibility of zero elements. Leslie’s method, that is, “choosing zero elements” to make a subset, seems to be more accessible for our participants than drawing the conclusion by applying the definition. Another pedagogical approach is suggested by the interviewer in the following dialogue. The student is led to explain that the empty set is a subset of any set using indirect method or approach by contradiction:

Interviewer: Look at the set \{2,4,7\}. Is it a subset of \{1, 2, 3, 4\}?
Kathy: No.
Interviewer: Why not?
Kathy: Because 7 is an element in the first set but 7 is not an element in the other set.
Interviewer: So how do you determine if something is not a subset of another set?
Kathy: You find an element in the one set that is not an element in the other.
Interviewer: Now look at the empty set. Can you find an element in the empty set that is not in our second set?
Kathy: No.
Interviewer: So what can you say if it’s not not a subset?
Kathy: It’s a subset.

Kathy claims that in order to infer that Set B is not a subset of Set A, it is sufficient to find an element \(x\) in B, which is not an element of A. For the empty set, such an \(x\) is not found, that is, there is no element in B, which is not in A. Therefore, it can’t be shown that the empty set is not a subset of A, which leads to affirmative conclusion that it is a subset of A. We would not recommend this approach as the first explanation to start with. However, it seems to prove beneficial as a reconfirmation of the fact that the empty set is a subset of every set at a more advanced stage.

Venn Diagrams, Container Images, and Visual Representations

Venn diagrams provide a visual representation of sets and are used to illustrate the relationship between sets in terms of being equal, inclusive, or disjoint, and also to represent intersection and union of sets. Research literature on visualization shows that visualization may be useful and helpful for the understanding of mathematical concepts, but may also be misleading.
Sets, Subsets, and the Empty Set

(Dreyfus, 1991). Venn diagrams are not an exception. Scherlock and Brand (1973a, 1973b) pointed out that Venn diagrams are at times used inaccurately when the universe of discourse is not shown. They claimed that the serious drawback in the use of Venn diagrams is that they are not systematic and are not easily extended when more than three sets are involved. Based on this argument, Scherlock and Brand suggested in their article “Venn Diagrams Must Go?” to replace the use of Venn diagrams by representations of sets as shaded areas on a rectangular grid regions, known as Karnaugh map or Veitch diagram. Dodridge (1973) pointed out the “topological” difficulties with Venn diagrams, such as labeling the sets inside or outside the diagram, but suggested that Karnaugh maps should be used as additional imagery, and not as a replacement to Venn diagrams.

Our investigation didn’t intend to focus on students’ use of Venn diagrams. Nevertheless, since they were one of the topics approached in the course, some students used Venn diagrams to explain their decisions on the written assessment and to exemplify their ideas during the interview. From students’ responses we find that the visualization provided by a Venn diagram may be interpreted inconsistently with the symbolic representation of sets, especially when some of the set elements are sets. Look for example at the Set $S=\{1,2,\{3,4\}\}$. $S$ has three elements: 1, 2, and the set $\{3,4\}$. Often to explain the nature of this set, instructors emphasize the “container” idea. One may think of the Set $S$ as a container, when a container $S$ is opened, one finds in it the number 1, the number 2, and another container, which in turn contains numbers 3 and 4. Regardless of whether the container idea is brought to students in words or pictures, the mental image of $S$ may be close to one shown in Figure 3, which is just a two-dimensional spread of the symbolic writing of the Set $S$ as one-liner.

![Container image of the Set $S=\{1,2,\{3,4\}\}$](image)
If \( T = \{3,4\} \), how can the Sets \( T \) and \( S \) be shown in a Venn diagram? Actually, \( T \) and \( S \) are disjoint, which is demonstrated in Figure 4(a). But having \( T \) as an element of \( S \) has often resulted in students’ suggestion for the Venn diagram for \( T \) and \( S \) as demonstrated in Figure 4(b). Further, the confusion between Venn diagrams and the container images may result in making consistently incorrect decisions regarding set elements and subsets. Laurie, for example, made on her written assessment a container diagram of the Set \( A \) (see Figure 5) and then referred to it as if it were a Venn diagram to determine the subsets.

![Figure 4. Venn diagrams for \( S = \{1,2,\{3,4\}\} \) and \( T = \{3,4\} \)](image)

(a) correct  (b) incorrect

![Figure 5. Laurie’s diagram for Set \( A = \{5,7,\{5\}, \{5,7,\{7\}\}\} \)](image)
Regardless of whether or not the container diagrams are used explicitly, they are implicit in the set structure. Venn diagrams are similar looking but represent a different perspective: Brackets within brackets (or a bag within a bag) represent set element; a circle within a circle represents a subset. We bring this to the attention of mathematics educators, since we believe that helping students to avoid the misconception starts with teachers’ awareness of the possible problem involved.

Venn diagrams, due to their flat structure, are not generalizable to represent sets with sets as their elements. According to Freudenthal (1973), “relations such as one set being an element of another, ...were never represented by Venn diagrams simply because Venn diagrams cannot do it” (p. 343). Venn diagrams are helpful to represent operations on sets, but not the set idea or structure.

**DISCUSSION**

**Historical Note**

An interesting historical aside comes from the history of transfinite numbers. Sfard (1991) argues that the development of mathematical concepts historically has parallels with the development of mathematical concepts cognitively by individuals. We believe a historical analog can be drawn from the concept of an infinite set as used by Cantor (cited in Dauben, 1979) in his proof on the denumerability of the real numbers.

In much the same way that 5 can be viewed as the process 1, 2, 3, 4, 5 (i.e., counting 5), the concept of infinity (aleph 0) can be viewed as the process of counting without an end. One of the unique features of Cantor’s proof of the denumerability of the set of real numbers is that he treats infinite sets as objects. Cantor has suggested that collections of objects were to be treated as wholes. Dauben, in his synopsis of Cantor’s work, writes:

The Beiträge’s² first sentence is a classic. It set the tone for all that was to follow: “Definition: By a ‘set’ we mean any collection M into a whole of definite, distinct objects m (which are called the elements of M) of our perception or of our thought.”
It is significant that Cantor even bothered to define the concept of set at all... In his own words: By an “aggregate” or “set” I mean generally any multitude which can be thought of as a whole, that is, any collection of definite elements which can be united by a law into a whole. He was primarily interested in sets as a whole because only in such terms could the transfinite numbers be defined. (Dauben, 1979, p. 170)

In 1874, Cantor showed that any listing of the real numbers is going to be incomplete. To do this he assumed that if such a list existed, one could form an infinite collection of elements. These elements were themselves to be infinite sets. So essentially one gets an infinite set of infinite sets. The elements were arrived at through a process of choosing elements from the original list with the property of being the next highest. So in some sense there was still a lingering perceptual view. In the 1891 diagonalization proof, Cantor just assumed infinite sets as elements E1, E2, E3, ...thereby skipping the process almost entirely. This proof seems to mark historically the “encapsulation” of the infinite set. It appears that in the history of set theory one of the greatest shifts in the concept of infinity, that is, existence of higher orders of infinity, was achieved with the encapsulation of the process of counting without end. In the next section we summarize some developmental observations consistent with the findings of this research.

**Objects and Collections of Objects**

Encapsulation of an (infinite) set concept, that is, thinking of the set and treating it as one entity, was crucial in the development of set theory. Analogously, encapsulation of a (finite, to start with) set concept is crucial for a student in her or his learning of set theory. The construction of the set concept starts with explicitly putting the elements together in a set. “If students can think of one set as a legal element in another set, does this mean that they can think of a set itself as an object and not just a collection of objects?” asked Baxter (1994, p. 104). We believe that the answer is positive, or, at least, this ability is a step toward objectification of a set concept. In specific, a possible indication for encapsulation of a set as an object is when a set can be identified and used as an element of another set. When the set $S=\{1,2,\{3,4\}\}$ is identified as a set of three elements, where 3 and 4 are not the elements, but $\{3,4\}$ is, we see an indication that the set $\{3,4\}$ was treated as one conceptual entity, that is, it has been encapsulated to an object. However, we believe that the main difficulty students face is not in considering a set as one conceptual object but in considering a set as an object
and as a collection of objects simultaneously. The obstacle here seems to be
similar to the one described by Piaget (1965) in his studies of additive
composition of classes, that is, inclusion of partial classes in a wider class.
Here is one of Piaget’s examples: “Juil watched me drawing 12 girls and 2
boys: ‘Are there more girls or more children in this class?—More girls.—
But are the girls children?—Yes.—Then are there more children or more
girls?—More girls”’(p. 167). Juil is aware that girls are children but has
difficulty in including girls in both classes simultaneously. “The child ap-
parently forgets the whole when he thinks of the part and forgets the part
when he thinks of the whole” (p. 171). Piaget observed that a step toward
acquiring the concept of inclusion of classes is made by quantitative com-
parison. In the example above a child could see that there are 12 children
and 10 girls. The fact that the number indicating “how many children” is
bigger than the number indicating “how many girls” leads the child to the
conclusion that “there are more children than girls” some time before the
qualitative understanding of the inclusion of classes is acquired.

A child’s developmental discovery of additive composition of classes,
what mathematicians refer to as transitivity, may add an additional dimen-
sion of difficulty when the transitivity doesn’t apply. In our above example
of the Set S={1,2,{3,4}}, students’ repeating confusion is to consider 3 or
4 as elements of S, even when it is understood that {3,4} is an element of
S. We described this mistake in a previous section as overgeneralization
of transitivity. Based on Piaget’s experiments we suggest that quantitative ar-
guments might help students to construct the concept of an object belong-
ing to a set. Counting and listing the three elements of S could help to rea-
lise that 4 is not one of these three elements. This approach of quantitative
comparison may help students to consider {3,4} simultaneously as a set of
two elements on one hand, and as one of three objects/elements that belong
to S on the other hand.

Mathematically, 4 is not an element in Set S, where S={1,2,{3,4}}.
Neither is 5. However psychologically, while it is easy to accept the fact
that 5 doesn’t belong to S, there is a tendency to acknowledge relationship
between 4 and S: The number 4 is an element of an element. Pedagogically,
following the analogy “the son of your son isn’t your son, it is your
grandson,” we suggest to call 4 in S a “grand-element.” This analogy was
accepted by students with a great enthusiasm, however further research is
needed to find its value in understanding of set concepts.
Two Approaches to Subsets

Considering the construction of a subset as an action, it is useful to distinguish between two possible kinds of actions: (a) given S, making a subset out of the elements of S and (b) given S and B, checking whether B is a subset of S. The first action is performed by clustering together any number of elements of S (or alternatively, casting out any elements that are not in the subset). The second action is performed by considering individually each and every element of B and verifying that it was an element in S as well. When these actions can be carried out in the individual’s mind, without going explicitly through all the steps, the action of making a subset or checking for a subset is interiorized to become processes. In this case the learner might claim, for example, that B={10,11,12...20} is a subset of S={1,2,3...50} since all the elements of B were in S, without checking each element individually. The processes interiorized from both actions may be coordinated as follows: Given Sets S and B, a learner would check whether B is a subset of S by seeing whether B “can be made” from the elements of S. In this case the learner could claim that {40,50,60,70} was not a subset of S pointing out that an element 60 was not an element of S, without checking all the other elements. One possible indication of encapsulation of a subset to an object is when a learner is able to consider the power set of a given Set S, that is, to carry or to intend the action of “clustering” into a set all the subsets of S.

To consider the empty set as a subset of every set, we find that the first action approach, that is, choosing zero elements, is more powerful for beginners than the second action approach, that is, working out the definition.

SUMMARY AND CONCLUSION

In our investigation of preservice teachers’ understanding of some ideas related to sets—set cardinality, set element, subset, and the empty set—we found that often meanings unintended by mathematical conventions were assigned to these concepts. Our purpose in this study was not to show the low success rate, but rather to point out some unconventional construction that were robust among the participants.

A frequent confusion was interpreting the set element, which is itself a set, as a subset. Freudenthal (1969) pointed out that children look for meaningful definitions, not formal. Probably, the same observation can be made about elementary school teachers. When asked what a subset was,
they were trying to describe the meaning, not to recall the formality. The prefix “sub” carries with it the associations of beneath, under, subordinate, and secondary. It implies some kind of hierarchy that was interpreted by many of our participants as “a set within a set.” Also, probably relying on “everyday life” experiences, many participants interpreted the relation “element in the set” as a transitive one. The difference between the symbols \{发生的\} and \{\}, or even \{5\} and 5, was at times not recognized.

The idea of the empty set appeared to be a complex one in this group of preservice teachers. The complexity showed when some of the properties of the empty set, such as being a subset of every set, were considered. We noticed that the ISETL experience seemed to help students with acquiring the ideas of set element and a subset, and actually some participants explicitly acknowledged this in the interviews, but it had only a little effect on students’ construction of the empty set. Of course, in order to fully explore the influence of ISETL exposure on constructing mathematical concepts it would be necessary to make it an integral part of learning, not just an add-on project. It is possible that presenting the questions to the students in a context of problem situation (story problem) could have increased their understanding of sets. We leave this query for further investigation.

“A concept of set is a very general and a very simple one” (Krause, 1991, p. 28). General? Indeed. Very simple? Controversial. The above pages are the result of our attempts to demonstrate the complexity of the “very simple” set concept.

References


**Notes**

1. It is not necessary to define a separate func “findelement” in order to assign to \( x \) a value of arbitrary element of \( B \). We’ve chosen to do so for the clarity of the following discussion.

2. Beiträge is short for *Beiträge zur Begründung der transfiniten Mengenlehre*, that was Cantor’s last major mathematical publication, published in 1895.
Appendix A: List of Problems in the ISETL Project

1. There are at least 3 different ways to define in ISETL the set of numbers
\{5,10,15,20,25,30,35,40,45,50,55,60,65,70,75,80,85,90,95,100\}
What are they? Verify with ISETL that the sets you’ve defined are indeed equal.

2. There are at least 3 different ways to define in ISETL the set of numbers:
\{2,9,16,23,30,37,44,51,58,65\}
What are they? Verify with ISETL that the sets you’ve defined are indeed equal.

3. What are the following sets? Explain in words.
   SET1:= \{3*x | x in \{1..20\}\};
   SET2:= \{645, 2*645..10000\};
   SET3:= \{k | k in \{1,3..100\} | k mod 4 = 0\};
   SET4:= \{n | n in \{100..200\} | (n mod 11 = 0) and even (n)\};

4. What is the cardinality (# of elements) of each one of the following sets? Check your prediction with ISETL. Explain.
   (a) K1:= \{“isetl”, “logo”, 15, \{3*5, \{1..10\}\}, \{}, 1993\};
   (b) K2:= \{5, 2+3, “4+1”, \{5\}\};
   (c) K3:= \{5>3, 3>5, 2+2=4, 9 mod 3 = 0, \{5>8\}\};

5. Express #(S union T) in terms of #S, #T and #(S inter T)
   Find other possible connections among #S, #T, #(S union T), #(S-T), #(T-S) and #(S inter T).
   Check your findings with ISETL on a couple of different examples.

6. Define the following sets:
   (a) the set of the cubes of integers between 0 and 10
   (b) the set of all the integer numbers from 100 to 150
   (c) the set of the even numbers between 0 and 100 that are also divisible by 7
   (d) the set of the odd numbers between 0 and 100 that are also divisible by 8
   (e) the set of all the integer numbers between 0 and 100
   (f) the set of all the numbers between 100 and 200 that have a remainder of 2 when divided by 7

7. The greatest common divisor (G.C.D) of two whole numbers a and b is the largest whole number that divides both a and b.
   Define: the set of all the divisors of number 100
   the set of all the divisors of number 64
   the set of common divisors of 100 and 64
   the greatest common divisor of 100 and 64
   the greatest common divisor of 72 and 240

8. The lowest common multiple (L.C.M.) of two whole numbers a and b is the smallest whole number which is a multiple of both a and b.
   Define: the set of multiples of 75 (with a reasonable upper limit)
   the set of multiples of 120 (with a reasonable upper limit)
   the set of common multiples of 75 and 120
   the lowest common multiple of 75 and 120
   the lowest common multiple of 35 and 231
9. Consider the empty set. Write a paragraph about the empty set and its properties. Demonstrate with examples using the ISETL code.

10. (a) Suppose the operation inter (intersection of sets) is not defined in ISETL. How would you define in ISETL a set I which is the intersection of A and B (without using “inter”)? Demonstrate your solution on several examples.

(b) Suppose the operation “compliment of a set” (A-B) is not defined in ISETL. How would you define a set C which is a compliment of B in A? Demonstrate your solution with several examples.

11. Write an ISETL code that will help you to check whether the numbers 1097 and 1099 are prime numbers. EXPLAIN.

12. Pythagorean Triple is a set of three whole numbers {a,b,c} such that a²+b²=c². Use ISETL to find all the Pythagorean triples among numbers smaller than 20.