Organizational Techniques for C:M Ratios in Frequency Modulation

Barry Truax
Department of Communication Studies
Simon Fraser University
Burnaby, B.C., Canada

1. INTRODUCTION

Since the introduction of the frequency modulation technique of timbral synthesis (Chowning, 1973), a considerable amount of interest has been expressed in this method, and a large amount of work has been done in realizing it in a variety of hardware and software configurations. Little, however, appears to have been published as to the systematic organization of one of the main variables of the technique, namely the ratio between the carrier frequency *c* and the modulating frequency *m*, also called simply the *c:m* ratio. The importance of this variable will become clear when it is remembered that it controls the set of partials (called sidebands) in the resultant spectrum. In general, a wide variety of combinations of harmonic and inharmonic partials may be generated depending on careful choice of the *c:m* ratio. Although it is probably not possible to generate any arbitrary set of partials with this method (for instance, because the partials are always separated by the same linear frequency difference from each other), any useful method of relating *c:m* ratios will presumably enhance compositional control over their use. Moreover, since some newer synthesis methods, in particular those of Andy Moorer and Marc LeBrun, coming from Stanford, take FM as a model or starting point, the applicability of *c:m* ratio properties may very well transfer to future techniques.

The basic problem of the organization of partials through the sideband generation method of FM stems from the basic form of the FM spectrum, namely:

$$|c \pm nm|, \quad n = 0,1,2,\ldots$$

In this form, n represents the sideband pair number, the + and − the upper and lower sidebands respectively, and the absolute value signs, the fact that lower sidebands which become negative are reflected into the positive frequency domain with a phase shift of 180°. Thus we have a set of symmetrically spaced partials radiating outward from the carrier frequency. The reflected lower sidebands, in some sense, create a lot of the interest in the spectrum, since these may or may not fall on the same frequencies as some of the uppper sidebands. The condition for them to coincide is that the *c:m* ratio must be of the form N:2, where N is a positive integer; in all cases of N the spectrum consists of some subset of the harmonic series. With all other ratios, the lower sidebands do not fall against the upper ones, and a considerably richer spectrum results.

Degree of Harmonicity

One generally distinguishes the type of spectrum produced by a *c:m* ratio as being more or less harmonic, that is, by how many sidebands are multiples of the fundamental, and how many are not. To discuss this properly, we need to clarify the means of expressing the *c:m* ratio. Most conveniently this ratio is expressed as a pair of integers. Real numbers could also be used, but since these can be approximated closely by some integer pair, we will keep to integers and use their properties. The most intuitive sense of relating degree of harmonicity to *c:m* ratio is that the simpler the ratio, the more harmonic the spectrum, and vice versa.

A more precise way of saying the same thing is that the lower the least common multiple of the two integers *c* and *m*, the greater the incidence of sidebands related harmonically to the fundamental. For instance, the extreme end of harmonicity is the 1:1 ratio which produces the entire harmonic spectrum. Likewise, all ratios of the form N:1 reproduce this same spectrum with the carrier simply becoming the Nth harmonic in the harmonic spectrum. Next come ratios such as 2:3 and 3:2 where the carrier is above the fundamental and all sidebands are multiples of it. This may be generalized for all ratios of the form N:N+1 or N+1:N, but although such ratios produce a subset of the harmonic spectrum, as N gets higher, one progressively loses the sense of harmonicity, just as with ratios of the form 1:N.

After these ratios producing harmonic spectra come all other ratios where at least some sidebands are inharmonic. Examples using the smallest integers are the ratios 2:5 and 3:5 which produce sidebands in the series (2 3 7 8 12 13 . . .). This may be compared to the "missing fundamental" case where the fundamental is "filled in" by the higher order auditory processing if it is strongly enough "suggested" by harmonics, such as 2 and 3. Theoretically, since we are only using integers, then *any* ratio produces some set of integer-related sidebands above some theoretical, absent "fundamental." However, as the brain is forced to rely on higher and higher harmonics to supply a fundamental, the effect breaks down.

It should also be remembered that the nth sideband pair is a multiple of the carrier *only* when n equals c or is some multiple of it. As c increases from small to large integers, then the possibility of hearing harmonic relations of sidebands to carrier frequency falls off proportionally.

Typically, then, ratios such as 5:7 produce what can be heard as distinctly inharmonic spectra (the sideband series is 2 5 9 12 16 19 23 26 ...), comparing favorably with Chowning's ratio 1:1.414... used to produce an inharmonic bell-like spectrum. This occurs also in the integer case despite the presence of a 6th and 8th harmonic.

Compositional Ordering of C:M Ratios

Although the above relationships may suggest some kinds of compositional ordering of the spectra of certain $c{:}m$ ratios according to their degree of harmonicity, the tendency is to use these various properties as "rules of thumb," as opposed to systematic principles. It should be noted here, however, that we are assuming that it is somehow going to be desirable to use *different* spectra in the same composition. Constant timbre, on the other hand, can only be achieved through $c{:}m$ ratios that produce the same spectrum. It is when we wish to organize different spectra in a systematic way that the rest of this paper becomes of any use.

Two methods of organizing $c{:}m$ ratios will be presented. These are: 1) By predicting the precise interval between the carrier and the actual fundamental, and relating that interval to just or tempered scales; 2) By predicting sets of $c{:}m$ ratios producing unique spectra and those producing exactly the *same* spectrum (i.e. the same set of sidebands).

These methods arose naturally through the use of the author's POD6 program for synthesis and composition because this program has the property of being able to select either the carrier or the modulating frequency *independently* of the $c{:}m$ ratio. This seemingly unorthodox approach led to the techniques for, first, keeping a constant carrier frequency and determining fundamentals via $c{:}m$ ratios, and secondly, keeping the modulating frequency constant and using the set of $c{:}m$ ratios that produce the same spectrum, in contrast with other sets producing a different spectrum. These two methods derived from compositional practice led to the systemization summarized by 1) and 2) above and elaborated in the rest of this paper.

II. ORGANIZING C:M RATIOS BY PREDICTING THE FUNDAMENTAL

As will be clear to anyone who has worked even briefly with the frequency modulation synthesis technique, the carrier frequency is not always the fundamental frequency of a complex of sidebands produced with a given $c{:}m$ ratio. In some compositional instances, one wishes to predict, perhaps even control very precisely, where the fundamental frequency lies for any given $c{:}m$ ratio. We may first distinguish two cases:

1. The carrier frequency is always the fundamental for a $c{:}m$ ratio where $m \geqslant 2c$, i.e. for the $c{:}m$ ratio 1:2 and for those ratios where $m{:}c$ is greater than 2.
2. All other cases in which the fundamental is below the carrier and is either:

   a) the lowest unreflected lower sideband, or:

   b) the first reflected lower sideband.

In some cases, 2a) and 2b) are identical, but in most it will be a situation where either 2a) or 2b) will be true, that is, one of the lower sidebands closest to 0 Hz. will be the fundamental.

The only other cases are two special instances of 2., namely the harmonic ratios 1:1 and 0:1 which produce all harmonic sidebands above the carrier in the case of 1:1, and above the first pair of sidebands for 0:1. These will be ignored in the following discussion.

To predict $c{:}m$ ratios which have a given fundamental (as in 2. above) we follow this argument: Let the ratio of the carrier to the fundamental be expressed as a/b. Note that this allows both just scale intervallic notation, e.g. 15/8 or equal tempered notation, e.g. 2/1.887749.

Case 2a): The lowest unreflected sideband is the fundamental.

Therefore: $$\frac{c}{c - nm} = \frac{a}{b} \qquad \text{for } n = 1, 2, 3, \ldots \qquad (1)$$

Given a ratio a/b, we may find the appropriate $c{:}m$ ratio(s) by expressing (1) as:

$$\frac{c}{m} = \frac{na}{a - b} \qquad \text{for } n = 1, 2, 3, \ldots \qquad (2)$$

However, although the given pitch will be present for all such $c{:}m$ ratios determined in this way, equation (2) does not guarantee that the desired frequency will be the lowest. Only certain values of n in equation (2) will produce the desired fundamental. Since the sidebands are spaced apart by m, the pair around 0 Hz. such that one is positive and the other negative will determine which of those two becomes the fundamental. The condition will be that the absolute value of the sideband which is the fundamental will be less than or equal to one-half the value of m. In the case of this distance being exactly one-half m, then both sidebands will fall on the same frequency and be the fundamental.

The condition just described can be expressed as:

$$c - nm \leqslant m/2$$

Therefore: $$n \geqslant \frac{c}{m} - \frac{1}{2}$$

Substituting from equation (2):

$$n \geqslant \frac{na}{a - b} - \frac{1}{2}$$

Solving for n:

$$n \leqslant \frac{a - b}{2b} \qquad (3)$$

Note that the minimum value of n should be 1. Therefore $(a-b)/2b$ must be greater than or equal to 1. Solving for a/b we find that a/b is greater than or equal to 3. This shows that for this case, only intervals that are greater than or equal to 3/1 downwards from the carrier are obtainable for a fundamental.

Case 2b): The first reflected sideband is the fundamental.

Following the same approach, we may conclude that (similar to (1)),

$$\frac{c}{nm - c} = \frac{a}{b} \qquad (4)$$

Therefore:

$$\frac{c}{m} = \frac{na}{a+b} \qquad (5)$$

The condition for the fundamental, again, is:

$$nm - c \leqslant m/2$$

Therefore:

$$n \leqslant \frac{c}{m} + \frac{1}{2}$$

Substituting from (5), we arrive at:

$$n \leqslant \frac{a+b}{2b} \qquad (6)$$

Again, since n is minimally 1, $(a+b)/2b$ must be greater than or equal to 1, and therefore (a/b) must be greater than or equal to 1. In other words, *any* fundamental frequency below the carrier is obtainable by a ratio where the reflected sideband produces the required frequency, whereas above in case 2a), only those frequencies lower than 3/1 below the carrier were possible.

Summary

In all cases where the fundamental is below the carrier and forms a ratio of a/b downwards from the carrier, the c:m ratio which produces that fundamental may be found from the relations:

case a): the lowest unreflected sideband is the fundamental:

$$\frac{c}{m} = \frac{na}{a+b} \quad \text{for all } n \text{ where } \quad n \leqslant \frac{a+b}{2b}$$

case b): the first reflected sideband is the fundamental:

$$\frac{c}{m} = \frac{na}{a-b} \quad \text{for all } n \text{ where } \quad n \leqslant \frac{a-b}{2b}$$

Note that the value of n gives the number of the lower sideband which becomes the fundamental, or the lowest pitch of the resultant spectrum. In terms of frequency modulation, an approximate rule of thumb is that the modulation index is approximately equal to the number of sideband pairs which have a significant amount of energy in the resultant spectrum. This means that a modulation index of I about equal to n will be necessary to actually have this fundamental heard in the spectrum. In addition, the carrier frequency must be in a sufficiently high range such that the fundamental is audible. Naturally for a fundamental very far below the carrier, its presence may be perceived more as a low rumble than a true fundamental. Therefore, it has been assumed in this argument that the desired fundamental lies in a normal musical range (above 60 Hz.).

Computer Calculation

A computer program was written to calculate the c:m ratio that best produced the desired fundamental below the carrier as expressed by the interval a/b. The two cases above were both considered. The fundamentals chosen were those pitches that figure in the major, minor and chromatic just scales, and those from the equal-tempered scale. Six octaves for each pitch were considered, that is, each instance of the desired pitch in the six octaves below the carrier. Although intervals are normally calculated or expressed as being above a given note, e.g. 3/2 being the fifth above, these ratios were simply converted into downward intervals, such as 4/3 for the downward fifth, or 8/3 for the downward 12th, and so on.

For many different reasons, c:m ratios are best handled as an integer ratio, and therefore the goal of the program was to find the "best fitting" integer c:m ratio. In addition, the current POD6 limitation that the c and m ratio values must be less than or equal to 255 was observed. In some cases, this prevented the most accurate ratio being found, but this only occurred for fundamentals four or more octaves below the carrier. At least one accurate ratio exists for all six octaves, and in most cases, a multiplicity of ratios (through the various possible n values) were found which produced the required fundamental within 2 to 5 cents. In the first three octaves below the carrier, the error was less than 0.5 cents; the results for these three octaves are reported in Tables I and II for the just and equal tempered scales respectively.

III. ORGANIZING C:M RATIOS ACCORDING TO SPECTRAL IDENTITY AND UNIQUENESS

Instead of organizing c:m ratios by predicting a carrier/fundamental interval, we now turn to examining the spectra, or sideband sets, produced by certain c:m ratios. In particular, we wish to know which c:m ratios produce identical spectra, i.e. the same set of sidebands, and which produce spectra that are entirely unique, i.e. non-identical sets of sidebands.

In the case of identical spectra or sidebands, the answer is readily obtained. For the same sidebands to be generated by two c:m ratios, their m values clearly must be identical; what is then different between the spectra is simply the matter of which partial is the carrier frequency. For any sideband set, a c:m ratio exists where the carrier frequency can become the frequency of any sideband.

To clarify the means of expressing c:m ratios, we will refer to the *normal form* of a c:m ratio, by which we mean the case where the carrier is the fundamental, or more precisely, where $m \geqslant 2c$. This is the first case considered in the previous

Upward Interval	Downward Interval	NOTE	C:M RATIOS					
			1st Octave Below		2nd Octave Below		3rd Octave Below	
			Unreflected	Reflected	Unreflected	Reflected	Unreflected	Reflected
2:1	1:1	C	–	1:2	–	2:3	4:3	4:5 8:5
15:8	16:15	B	–	16:31	–	32:47	64:49	64:79 128:79
9:5	10:9	B♭	–	10:19	–	20:29	40:31	40:49 80:49
7:4*	8:7	B♭⁻	–	8:15	–	16:23	32:25	32:29 64:39
5:3	6:5	A	–	6:11	–	12:17	24:19	24:29 48:29
8:5	5:4	A♭	–	5:9	–	5:7	5:4 5:2	5:6 5:3 5:2
3:2	4:3	G	–	4:7	–	8:11	16:13 32:13	16:19 32:19 48:19
36:25	25:18	G♭	–	25:43	–	25:34	50:41 100:41	50:59 100:59 150:59
45:32	64:45	F♯	–	64:109	–	128:173	165:136 165:68	165:194 165:97 199:78
4:3	3:2	F	–	3:5	3:2	3:4 3:2	6:5 12:5	6:7 12:7 18:7
5:4	8:5	E	–	8:13	16:11	16:21 32:21	32:27 64:27	32:37 64:37 96:37
6:5	5:3	E♭	–	5:8	10:7	10:13 20:13	20:17 40:17	20:23 40:23 60:23
9:8	16:9	D	–	16:25	32:23	32:41 64:41	64:55 128:55 192:55	64:73 128:73 192:73 249:71
16:15	15:8	D♭	–	15:23	15:11	15:19 30:19	15:13 30:13 45:13	15:17 30:17 45:17 60:17
1:1	2:1	C	–	2:3	4:3	4:5 8:5	8:7 16:7 24:7	8:9 16:9 8:3 32:9

*The so-called harmonic seventh interval is included for comparison even though it does not appear in the major or minor just scales.

Table I.: C:M Ratios producing Just Scale Intervals between the Carrier and the Fundamental.

INTERVAL	NOTE	C:M RATIOS 1st Octave Below Unreflected	1st Octave Below Reflected	2nd Octave Below Unreflected	2nd Octave Below Reflected	3rd Octave Below Unreflected	3rd Octave Below Reflected
1.887749	B	–	107:208	–	89:131	89:68	89:110 89:55
1.781797	B♭/A♯	–	55:104	–	110:159	220:171	202:247 229:140
1.681793	A	–	44:81	–	88:125	176:139	176:213 195:118
1.587401	A♭/G♯	–	63:113	–	63:88	126:101 252:101	126:151 252:151 253:101
1.498307	G	–	4:7	–	179:246	251:204 251:102	203:241 251:149 235:93
1.414214	G♭/F♯	–	140:239	–	99:134	198:163 243:100	198:233 243:143 181:71
1.334840	F	–	3:5	3:2	3:4 3:2	6:5 12:5	6:7 12:7 18:7
1.259921	E	–	127:207	127:87	127:167 254:167	127:107 254:107	127:147 254:147 127:49
1.189207	E♭/D♯	–	37:59	37:26	37:48 37:24	74:63 148:63	74:85 148:85 222:85
1.122462	D	–	98:153	196:141	196:251 253:162	221:190 221:95 171:49	221:252 221:126 221:84 221:63
1.059463	D♭/C♯	–	151:231	185:136	185:234 185:117	219:190 219:95 204:59	219:248 219:124 151:57 219:62
1.000000	C	–	2:3	4:3	4:5 8:5	8:7 16:7 24:7	8:9 16:9 8:3 32:9

Table II.: C:M Ratios producing Equal Tempered Intervals between the Carrier and the Fundamental.

0:1	1:32	1:31	1:30	1:29	1:28	1:27	1:26
1:25	1:24	1:23	1:22	1:21	1:20	1:19	1:18
1:17	1:16	2:31	1:15	2:29	1:14	2:27	1:13
2:25	1:12	2:23	1:11	3:32	2:21	3:31	1:10
3:29	2:19	3:28	1:9	3:26	2:17	3:25	1:8
4:31	3:23	2:15	3:22	4:29	1:7	4:27	3:20
5:33	2:13	5:32	3:19	4:25	5:31	1:6	5:29
4:23	3:17	5:28	2:11	5:27	3:16	4:21	5:26
6:31	1:5	6:29	5:24	4:19	3:14	5:23	7:32
2:9	7:31	5:22	3:13	7:30	4:17	5:21	6:25
7:29	1:4	8:31	7:27	6:23	5:19	4:15	7:26
3:11	8:29	5:18	7:25	9:32	2:7	9:31	7:24
5:17	8:27	3:10	7:23	4:13	9:29	5:16	6:19
7:22	8:25	9:28	10:31	1:3	11:32	10:29	9:26
8:23	7:20	6:17	11:31	5:14	9:25	4:11	11:30
7:19	10:27	3:8	11:29	8:21	5:13	12:31	7:18
9:23	11:28	2:5	13:32	11:27	9:22	7:17	12:29
5:12	13:31	8:19	11:26	3:7	13:30	10:23	7:16
11:25	4:9	13:29	9:20	14:31	5:11	11:24	6:13
13:28	7:15	15:32	8:17	9:19	10:21	11:23	12:25
13:27	14:29	15:31	1:2				

Table III.: The c:m Ratio Series of Order 32.

section. Note that any $c{:}m$ ratio may be reduced to its normal form by successively applying the operation $c = |c - m|$ until the ratio falls into the proper range, e.g. 11:5 becomes $|11 - 5| : 5 = 6{:}5$, and then $|6 - 5| : 5 = 1{:}5$, where 1:5 is the normal form. Likewise 4:5 also has the normal form $|4 - 5| : 5 = 1{:}5$.

Rule 1: Given a $c{:}m$ ratio in its normal form, the set of $c{:}m$ ratios producing identical spectra or sidebands sets is:

$$|c \pm nm| : m \quad \text{for } n = 1, 2, 3, \ldots$$

Note that all such spectra have the same modulating frequency; however, not all ratios with the same modulating frequency produce the same spectra (compare: 1:5 and 2:5).

For example, the ratio 2:5, which is in its normal form, has identical spectra with 3:5, 7:5, 8:5, 12:5, 13:5, 17:5, 18:5, . . . In practice there will be a slight difference in timbre because the amplitude of each partial will be different for a given modulation index, even though the same set of partials is present. When two such spectra lie on the same fundamental, they will completely fuse perceptually.

To organize $c{:}m$ ratios according to the uniqueness of their spectra, we must turn to the mathematical series known as the *Farey series*, a conclusion for which the author is indebted to Eric Regener of Montreal. The Farey series and its properties are succinctly presented by Regener in his book as follows:

"The Farey series of order n is the ascending series of irreducible fractions between 0 and 1 whose denominators do not exceed n. The Farey series of order 7, for example, is

$$\frac{0}{1}\ \frac{1}{7}\ \frac{1}{6}\ \frac{1}{5}\ \frac{1}{4}\ \frac{2}{7}\ \frac{1}{3}\ \frac{2}{5}\ \frac{3}{7}\ \frac{1}{2}\ \frac{4}{7}\ \frac{3}{5}\ \frac{2}{3}\ \frac{5}{7}\ \frac{3}{4}\ \frac{4}{5}\ \frac{5}{6}\ \frac{6}{7}\ \frac{1}{1}$$

If a_1/b_1 is any term of a Farey series and a_2/b_2 its successor, the series has the property that

$$a_1 b_2 - a_2 b_1 = -1.$$

Moreover, any two fractions having this property are adjacent in the Farey series whose order is the greater of the two denominators."

Although the Farey series is defined over the range 0 to 1, it is symmetrical about 1/2, and from the definition given above of normal form, we can see that we only need the ratios between 0 and 1/2 where we equate the $c{:}m$ ratio to the fractions in the Farey series.

Rule 2: Given $c{:}m$ ratios in their normal form, those producing unique spectra belong to the fractions of a Farey series from 0 to 1/2 of order n where n is equal to the largest value of m.

Recalling that the Farey series of order n is that where the denominators of the fractions do not exceed n, we can define analogously a $c{:}m$ series of order n where the values of m do not exceed n. For instance, the $c{:}m$ series of order 9 is:

$$\frac{1}{9}\ \frac{1}{8}\ \frac{1}{7}\ \frac{1}{6}\ \frac{1}{5}\ \frac{2}{9}\ \frac{1}{4}\ \frac{2}{7}\ \frac{1}{3}\ \frac{3}{8}\ \frac{2}{5}\ \frac{3}{7}\ \frac{4}{9}\ \frac{1}{2}$$

The beginning of the series at the left could be completed with the harmonic ratios 0:1 or 1:1. Hearing this series on a low-pitched fundamental, one hears a progression of spectra from widely-spaced partials giving a high-pitched coloration to the sound (the ratio 1:9), gradually descending to the equally spaced sidebands (all odd harmonics) of the 1:2 ratio. This series was the structural basis of the tape part of the author's *Sonic Landscape No. 4 (1977)*, for organ and tape, where each normal form $c{:}m$ ratio, and its family of ratios producing identical spectra, formed a unique harmonic region in each layer of the piece.

Naturally, the series in this simple order with all the ratios in normal form tends to produce predictable results which would seldom be desirable compositionally. However, the purpose of this method is merely to create an organizational scheme which allows *all* $c{:}m$ ratios to find their place.

Having achieved that, one may then proceed to choose ratios by whatever means one wishes.

In terms of the remarks on harmonicity made earlier, it should be noted that the above series includes both harmonic spectra of the form $1{:}N$, and inharmonic spectra, characterized by $c \neq 1$. The first inharmonic spectra arise with the $c{:}m$ series of order 5, namely with the 2:5 ratio. In the series quoted above of order 9, 6 ratios are inharmonic and 8 are harmonic; however, the inharmonic ones are clustered towards the right end of the set, and the harmonic ones towards the left end, a pattern which the author found suggestive in the above-mentioned work.

The ratios of the $c{:}m$ series of order 32 are given in Table III.

IV. CONCLUSION

Frequency modulation is a powerful method for timbral synthesis; however, its usefulness remains limited by one's knowledge of the types of spectra produced by this method. To produce a constant timbre with FM, one simply maintains a single $c{:}m$ ratio. However, to organize different timbres requires some method of organizing $c{:}m$ ratios if one wishes to do it systematically. Clearly a wide variety of such methods could be generated. Even more elaborate ways than those presented here are possible, such as organization by interval content, or with respect to critical bandwidths. Two suggestive methods were reviewed here, namely an organization based on establishing a particular ratio between the carrier and fundamental frequencies, and one based on classifying sets of ratios into those producing spectra which are identical, and those producing an ordered set of unique spectra. Both methods have proved compositionally useful in experience with the author's POD6 program for composition and sound synthesis, and their strength lies in the fact that they are systemizations of empirical methods derived through use of an interactive computer system for composition.

REFERENCES

Chowning, John. "The Synthesis of Complex Audio Spectra by Means of Frequency Modulation." *J. Audio Eng. Soc.*, 21(7):526-534, 1973. Reprinted in *Computer Music Journal*, I(2):46-54, 1977.

Regener, Eric. *Pitch Notation and Equal Temperament: A Formal Study*. University of California Press, Occasional Papers No. 6, 1973.

Truax, Barry. "A Communicational Approach to Computer Sound Programs." *Journal of Music Theory*, 20(2), 1976. Reprinted in part in *Computer Music Journal*, I(3):30-39, 1977.