Example of computing a competitive equilibrium in an exchange economy

Problem:
Suppose there are only two goods (bananas and fish) and 2 consumers (Annie and Ben) in an exchange economy. Annie has a utility function $u_A(b, f) = b^2 f$ where $b$ is the amount of bananas she eats and $f$ is the amount of fish she eats. Annie has an endowment of $w_{bA} = 7$ bananas and $w_{fA} = 3$ kilos of fish. Ben has a utility function $u_B(b, f) = 2f + 3\log(b)$ and endowments of $w_{bB} = 0$ bananas and $w_{fB} = 4$ kilos of fish. Assume the price of bananas is 1.

(a) Write down the definition of a competitive equilibrium for this economy.

(b) Solve for the competitive equilibrium fish price $p^*_f$ and the competitive equilibrium allocation of fish and bananas between Annie and Ben.

Solution:
(a) The definition was given twice in class.
A competitive equilibrium is an allocation (four quantities) $(f^*_A, f^*_B, b^*_A, b^*_B)$ and prices $(p^*_b, p^*_f)$ (we will use $p^*_b = 1$ later) such that:
(i) given the prices, consumers maximize their utility at the allocation $(f^*_A, f^*_B, b^*_A, b^*_B)$ (i.e., these quantities are their consumer demands given the prices and the endowments)
(ii) markets clear, i.e. demand equals supply for each good:

$$f^*_A + f^*_B = w^f_A + w^f_B$$
$$b^*_A + b^*_B = w^b_A + w^b_B$$

(b) Set $p^*_b = 1$. Then proceed in the usual way – solve each consumer’s problem of maximizing utility subject to her budget constraint. That will give us his demand for each good as a function of her income and the prices. For person A we must solve:

$$\max_{b, f} b^2 f$$
$$\text{s.t. } b + p_f f = m_A$$

Proceed as usual (remember ch. 5-6). Take a monotonic transformation of the utility (log) and substitute $b$ from the BC to get a problem of one variable:

$$\max_f 2 \ln(m_A - p_f f) + \ln f$$

Take the derivative and set to zero:

$$-\frac{2p_f}{(m_A - p_f f)} + \frac{1}{f} = 0$$
cross-multiply and solve for the demand for fish by A:

\[ f_A^* = \frac{m_A}{3p_f} \]

(notice this is the usual formula for Cobb-Douglas demand that I used directly in class, however I wanted the full derivation here!). From the budget constraint, find A’s demand for bananas:

\[ b_A^* = \frac{2m_A}{3} \]

Also, given the prices and the endowments we also know that:

\[ m_A = 7 + 3p_f \]

Proceed the same way for person B who has quasi-linear preferences. B’s consumer problem is:

\[
\max_{f,b} 2f + 3 \ln b \\
\text{s.t.} \quad b + pf = m_B
\]

plug in from the constraint into the utility function:

\[
\max_b \frac{2(m_B - b)}{p_f} + 3 \ln b
\]

Take the derivative and set to zero (be careful here because this is quasi-linear preferences, so if income is low enough the consumer will spend his all income on b – a corner solution).

\[-\frac{2}{p_f} + \frac{3}{b} = 0\]

or, \(b_B^* = \min\{m_B, \frac{3p_f}{2}\}\).

From the budget constraint: \(f_B^* = \max\{\frac{m_B}{p_f} - \frac{3}{2}, 0\}\). Given the prices and the endowments we also know that:

\[ m_B = 0 + 4p_f \]

Notice that this is always larger than \(\frac{3p_f}{2}\) so there always will be an interior optimum, i.e.

\[ b_B^* = \frac{3p_f}{2} \]

and

\[ f_B^* = \frac{m_B}{p_f} - \frac{3}{2} = \frac{4p_f}{p_f} - \frac{3}{2} = 2.5 \]

Now, use the definition for competitive equilibrium (CE) from part (a) and the demands you obtained above to solve for the CE price of fish \(p_f^*\) and then plug this price in the demand functions to obtain the CE allocation.
By Walras’ Law you can use only the market clearing condition for one of the markets (the other will automatically clear at the same prices). For example, take the market for bananas. Market clearing requires demand for bananas $b_A^* + b_B^*$ to equal supply of bananas in the economy: $w_A^b + w_B^b$. That is:

$$b_A^* + b_B^* = \frac{2(7 + 3p_f)}{3} + \frac{3p_f}{2} = w_A^b + w_B^b = 7$$

from where we find (solving the above equation for $p_f^*$) that the CE price is

$$p_f^* = \frac{2}{3}$$

You may check that the fish market also clears at this price, i.e., that $f_A^* + f_B^* = w_A^f + w_B^f$ (as Walras’ law claims). Finally (do not forget!) the CE allocation is obtained by plugging $p_f^*$ into the demands $b^*$ and $f^*$ obtained above. You should obtain:

$$b_A^* = 6, \ b_B^* = 1, \ f_A^* = 4.5 \text{ and } f_B^* = 2.5$$