Dynamic Optimal Insurance and Lack of Commitment☆

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Abstract

This paper analyzes dynamic risk-sharing contracts between profit-maximizing insurers and risk-averse agents who face idiosyncratic income uncertainty and may self-insure through savings. We study Markov-perfect insurance contracts in which neither party can commit beyond the current period. We show that the limited commitment assumption on the insurer's side is only restrictive when he is endowed with a rate of return advantage and the agent has sufficiently large initial assets. In such a case, the consumption profile is distorted relative to the first-best. In a Markov-perfect equilibrium, the agent's asset holdings determine his period outside option and are thus, an integral part of insurance contracts, unlike the case when the insurer can commit. Whether the parties can contract on the agent's savings decisions or not affects the agreement as long as the insurer makes positive profits.

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JEL classification: D11, E21.

1. Introduction

There exist many situations in which people do not pre-commit to long-term (e.g., lifetime) contracts or are in fact legally constrained to short-term contracts. After some time, it is usually the case that at least one party is free to rescind or request a change in the contract terms. For example, the majority of labor contracts are relatively short-term, employees cannot legally commit to never quit their job and employers can terminate or change contract terms at a cost depending on the jurisdiction; housing rental agreements are usually signed for no longer than a year with both sides able to terminate upon proper notice; various insurance, TV and phone service contracts possess similar features. In this paper, we study one such situation assuming that the parties cannot commit to a contract longer than one period.

Consider a long-term interaction between a risk-averse agent facing idiosyncratic income risk and a risk-neutral insurer. We analyze the best dynamic way for the insurer to extract profits from the agent, when the insurer has a saving technology with a (weakly) superior rate of return than that of the agent. It is clear that if both parties could commit to an infinitely-long contract at time zero, the efficient outcome involves the insurer taking hold of the agent's initial assets, investing them in the superior saving technology and appropriately compensating the agent with transfers over time.[1] The insurer's ability to commit to compensate the agent in the future, after taking his assets, and the agent's commitment to not walk away from the contract ex-post, are key for this first-best arrangement to work. If such commitment is absent, then

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[1] For example, if the agent is more impatient than the insurer, it is optimal to front-load agent's consumption.

the insurer faces a trade-off between rent-extraction and production efficiency and hence, the agent is left with some of his assets to be carried at the inferior rate of return. Since assets determine the agent's outside option, their extraction can occur only gradually, distorting the optimal front-loading of consumption. In addition, the fact that the agent can walk away at some point of the contractual relationship, limits how low his welfare can be driven in the long-run, which further reduces total surplus.

The time-profiles of agent's consumption and insurer's profits depend on the degree of market power of the insurer, which we allow to be anything between the two extremes of perfect competition and monopoly. Under perfect competition, expected profits are zero in net present value. If insurers lack commitment, free entry prevents them from earning positive expected period profits, which in turn implies they will not accept negative expected profits at any point in time. Thus, competitive insurers' expected profits are exactly zero and only period-by-period actuarially fair insurance is offered, with the agent's initial assets invested in the low-return technology until they are depleted. When the insurer has market power, the lack of commitment limits, but does not eliminate, his ability to extract profits efficiently.

To be more specific, suppose insurers can carry resources over time at the gross rate of return, R while agents have access to their own savings technology with fixed exogenous gross return, r, where $r \leq R$. At the beginning of each period, an insurer can offer any new contract he likes and the agent is free to accept or go to his outside option. The agent's outside option (one possibility used by many authors is going to autarky forever but we allow more general cases) depends on his current asset holdings and therefore evolves endogenously over time. We use the solution concept of Markov-perfect equilibrium (MPE), as formally described in Maskin and Tirole (2001), to characterize dynamic insurance contracts in our setting. Markov-perfect equilibria capture the idea that only current, payoff-relevant variables (here, income realizations and agent's assets) affect the terms of equilibrium contracts and fit the notion of "bygones are bygones"—characteristics which we consider natural in a lack of commitment environment.

Our main result is that the assumption that the insurer can only commit to a one-period, as opposed to an infinitely-long contract, matters only if both of the following conditions hold: (i) the two parties' rates of return differ ($R > r$) and (ii) the agent has sufficiently large initial assets. Intuitively, when the insurer's return is strictly higher than the agent's, there are gains from the insurer extracting the agent's assets upfront and carrying them over time at the superior rate of return. This is only possible, however, if the agent is appropriately compensated with promises of future consumption. For this, commitment by the insurer beyond the current period is indispensable. Lacking such commitment, we show that asset holdings by the agent become an integral part of Markov-perfect insurance contracts since they determine the value of the agent's outside option and hence the value of future transfers. In addition, the insurer's lack of commitment distorts the slope of the optimal consumption profile by introducing a "wedge" in the standard Euler equation whenever $r < R$.

In contrast, when either (i) agent and insurer face equal rates of return ($r = R$) or (ii) the agent begins with zero or sufficiently low assets, we show that focusing on Markov-perfect insurance contracts is not restrictive at all—the exact same consumption time path and welfare is achieved in a "one-sided commitment" contract in which the insurer pre-commits to an infinitely-long contract subject to a per-period participation constraint by the agent. In the latter contract, assets and promised utility are interchangeable instruments that can be used by the insurer to ensure the agent stays on. As a result, a multiplicity of asset paths is possible. This multiplicity is avoided in a MPE where, as already mentioned, agent's asset holdings are non-trivial feature of dynamic insurance which opens the way to calibrating versions of the model to data (see Karaivanov and Martin, 2011).

The agent's inability to commit beyond the current period, i.e., being free to leave the contract at the beginning of each period, avoids immiseration problems present in other papers in this literature. In the general case we consider, when the product of the insurer's return R and the agent's discount factor β is less than 1, the optimal consumption path in the first-best (full commitment to an infinitely-long contract by both sides) is downward-sloping towards zero. Thus, eventually the agent will be worse-off than his best alternative (autarky or any outside option that gives more than zero consumption forever). In contrast, in an MPE consumption converges to a strictly positive value in finite time since the agent can always walk away. Hence, with the agent unable to commit beyond the current period, as long as his initial assets are positive, Markov-perfect insurance contracts always differ from the first-best contract, independent of the parties' rates of return.[2]

Given that asset holdings by the agent are a key feature of Markov-perfect insurance contracts, we also study the role of asset contractibility by comparing the case of "contractible assets", when the insurer can fully control agent's savings, with the case of "non-contractible assets", in which the agent can privately decide on the amount of his savings but in a way observed by the insurer. In many situations, principals (governments, insurance companies, banks, etc.) may have information about agents' assets but, for legal or other reasons, are unable to directly control agents' savings choices.[3] In the dynamic insurance literature allowing agents to accumulate assets typically yields one of three results, depending on the particular information and contractibility assumptions made: assets play no role (when the insurer directly controls agent's consumption); assets eliminate the insurer's ability to smooth the agent's consumption beyond self-insurance (Allen, 1985, Cole and Kocherlakota, 2001); or the environment becomes highly intractable (Fernandes and Phelan, 2000; Doepke and Townsend, 2006). In contrast to the latter case, we show that using MPE results in simple dynamic programs with a single scalar state variable avoiding the "curse of dimensionality" when the principal cannot control agent's savings.

We show that, in our setting, asset contractibility does not matter in the first best—the same optimal contract results. In contrast, in a Markov-perfect insurance contract asset contractibility affects the contract terms in all cases except when there is a perfectly competitive insurance market with free entry by insurers. Intuitively, whenever the insurer has some market power (not necessarily being a monopolist) and can generate positive profits by insuring the agent, private asset accumulation by the agent gives him an instrument to counter the insurer by controlling his future outside option.

Unlike much of the previous literature on limited commitment (Thomas and Worrall, 1988, 1994; Ligon et al., 2002; Kocherlakota, 1996; Krueger and Uhlig, 2006 among many others)[4] we assume that agents cannot renege after observing their income realization, that is, within the period. Reneging on an insurance contract after high income is realized is the main reason why full insurance may not be possible (at all, or in the short run) in those settings. In contrast, we obtain full insurance immediately and all the time. The lack of commitment friction we study is thus not about incentives to renege *after* obtaining windfall income, but about being able to costlessly leave a contract *before* the period's uncertainty is resolved, should its terms become worse than one's outside option—not unlike in labor, health-insurance or similar contracts. The applications we have in mind are therefore not about agents acting opportunistically to obtain short-term gains from breaking a contract but about agents' freedom to continue or terminate a contractual relationship.

Our work is most closely related to Krueger and Uhlig (2006), Phelan (1995) and Kovrijnykh (2010). As in this paper, Krueger and Uhlig (2006) study dynamic risk-sharing between risk-neutral insurers and risk-averse agents who face idiosyncratic income uncertainty. Their model is most similar to our "one-sided commitment" setting—insurers can fully commit to an infinitely-long contract while agents can renege in each period and move to another insurer without any cost or delay. A key difference is that agents in their paper can renege *after* observing the income realization. Thus, their main question is whether intratemporal insurance can be provided given the agent's commitment problem and potential for opportunistic behavior. Krueger and Uhlig show that the answer depends crucially on the relative size of the discount factors of insurers and agents. Autarky, partial insurance, or even full insurance in the long run can be supported in equilibrium depending on the model parameters. Phelan (1995) studies dynamic risk-sharing with limited commitment in a competitive setting. His model is similar to ours in that the agent can renege on a contract before seeing his income realization. However, unlike here, the agent has to sit idle for a period before

[2]The only case in which these two contracts are equivalent is when the agent starts with zero assets *and* $\beta R = 1$ since in this case the first-best contract is a repeated one-period contract.

[3]Hereby we differ from the literature on optimal contracting with hidden savings (Allen, 1985; Cole and Kocherlakota, 2001 among others) which assumes that the principal has no ability to monitor the agent's assets. On the technical side, our assumption of observable savings helps us avoid dynamic adverse selection and the possible failure of the revelation principle with lack of commitment (Bester and Strausz, 2001), while still preserving the intertemporal implications of savings non-contractibility.

[4]Thomas and Worrall (1988) characterize firm-worker contracts in which each party can deviate to autarky forever. Thomas and Worrall (1994) study limited commitment in a foreign direct investment problem with risk of expropriation. Ligon et al. (2002) analyze risk-sharing among agents who can revert to self-insurance through storage at any time. Kocherlakota (1996) studies risk-sharing between agents who cannot commit not to revert to autarky forever at any period. He analyzes the set of subgame-perfect equilibria and shows that, if agents are sufficiently patient, there is no efficiency loss associated with the inability to commit.

re-signing with another insurer. Phelan assumes that agent's income is unobservable to the insurer and studies the implications of this asymmetric information problem on the level of insurance provided. Setting $\beta r = 1$, he shows that partial insurance can be achieved in equilibrium and that the long-run consumption distribution is non-degenerate.

In contrast to the Krueger-Uhlig and Phelan papers, the main question we study is not whether full insurance can be achieved or not (in our model it is achieved, always) but the implications of agents' and insurers' lack of commitment (across, not within periods) on the time path of consumption relative to the first-best and on private asset accumulation by the agent. Asset accumulation is not discussed by the above authors since the insurance contracts they study can be implemented through promised utility alone because of the insurers' full commitment ability. Additionally, our results cover the cases of non-competitive insurers and/or generalized outside option for the agent.

Kovrijnykh (2010) studies a borrower-lender relationship with probabilistic enforcement. In essence, each period, after receiving a payment from the borrower, the lender may get an opportunity to renege on (change) his promised investment. Kovrijnykh finds that social welfare in this environment—regardless of whether contracts can be signed for one or infinitely many periods—can be lower than social welfare in an environment in which the lender cannot commit to honor the agreement within the period. As in the limited commitment literature cited above, Kovrijnykh thus studies opportunistic behavior (by the lender, in her case). Her focus is on marginal increases in commitment power from no-commitment to partial commitment by varying the contract enforcement probability. Our setting is different—we assume perfect enforcement within the period and interpret lack of commitment as the inability to bind oneself to actions beyond the current period. We show that under certain conditions (e.g., equal rates of return on assets) an increase in the principal's commitment ability from one-period to infinitely-long yields equivalent welfare outcomes while otherwise full commitment dominates one-period commitment.[5]

More generally, our paper relates also to the work of Acemoglu et al. (2006) on optimal taxation with lack of commitment and the literature on "markets vs. mechanisms"—e.g., Bisin and Rampini (2006), Acemoglu et al. (2008) and Sleet and Yeltekin (2008)—who build various political economy models of governments with inability to commit and analyze to what extent they can improve upon private-information constrained incomplete markets.

The rest of the paper is organized as follows. Section 2 presents the environment and characterizes the first-best contract with full commitment by both sides. Section 3 relaxes the commitment assumption, characterizes the properties of Markov-perfect contracts and compares them to "one-sided commitment" contracts. Section 4 considers three natural extensions. First, we show that whether agent's assets are contractible or not affects the terms of Markov-perfect contracts unlike in the first-best. Second, we analyze the special case when insurance is provided in a perfectly competitive market with free entry. Third, we analyze the case when the agent and insurer bargain over the terms of the contract, which serves as an example of how to derive the agent's outside option we take as given in previous sections. Section 5 concludes. All proofs are in the Appendix.

2. The Model

2.1. Environment

Consider a long-lived risk-averse agent who maximizes expected discounted utility from consumption. His period utility is $u(c)$, with $u_c(c) > 0$, $u_{cc}(c) < 0$ and u satisfying Inada conditions.[6] The agent discounts future utility by factor $\beta \in (0, 1)$. He produces output y, which he can consume or save. Output is stochastic and equals y^i with probability π_i where $\pi_i \in (0, 1)$ for all $i = 1, \ldots, n$, $n \geq 2$, and $\sum_{i=1}^{n} \pi_i = 1$. Let $0 \leq y^1 < \ldots < y^n$.

[5] Sleet and Yeltekin (2006) consider lack of commitment by the principal in a dynamic private information economy. They show that optimal allocations when the principal and agent have the same discount factor are equivalent to those in an economy with fully-committed principals who discount the future less heavily than the agents. In contrast, allowing discount factors to differ or not, we demonstrate the importance of the two parties' rate of return on assets for the resulting outcomes with full vs. one-period commitment by the principal.

[6] Throughout the paper we use subscripts to denote partial derivatives and primes for next-period values.

Since the agent is risk-averse and output is risky, the agent would like to smooth consumption over output states and time. He starts with assets, $a \geq 0$ which can be carried over time via a savings/storage technology with fixed gross return $r > 0$. Let the set of feasible asset holdings be $\mathbb{A} = [0, \bar{a}]$ where $\bar{a} \in (0, \infty)$ and chosen sufficiently large so that it is not restrictive. If $y^1 = 0$ the lower bound on assets coincides with the "natural borrowing limit" (Aiyagari, 1994); if $y^1 > 0$, the lower bound on asset holdings can represent additional credit frictions in the environment. In terms of how dynamic contracts are affected by the frictions we study, our results are not affected by this particular assumption.

By standard arguments, the savings technology does not allow the agent to perfectly insure against the (n-dimensional) output state randomness, thus there is scope for additional insurance. Suppose there exists a risk-neutral, profit-maximizing insurer ("principal") who can provide such insurance. Throughout the paper, we assume that the insurer can costlessly observe output realizations y^i and agent's assets a (see footnote 3).

The insurer discounts future profits at gross rate $R > 0$. The parameter R has either a technological or preference interpretation. The special case $r = R$, for which several important results are derived, is interpreted as the insurer being able to carry resources intertemporally using the same technology as the agent. If instead $R = \beta^{-1}$, one can think of both contracting parties sharing the same discount factor—a standard assumption in the literature. In general, we allow R to take any value in between these bounds. In addition, we assume $r < \beta^{-1}$ so that agent's assets remain bounded, consistent with the definition of $\mathbb{A}$.

Assumption 1. *$0 < r \leq R \leq \beta^{-1}$ and $r < \beta^{-1}$.*

2.2. Agent's outside option

Let the agent's outside option when contracting with the insurer be given by the function $B(a)$. Our results do no rely on the specifics of the agent's outside option. By considering a general function $B(a)$ we can vary the split of the gains-from-trade between the insurer and the agent from the extreme case of a monopolistic insurer, where all the surplus goes to the insurer, up to (but not including) the case of perfect competition, where all gains go to the agent. The case of a perfectly competitive (or "benevolent") insurer is analyzed separately in Section 4.2, since it requires a different mathematical formulation. We make the following assumptions.

Assumption 2. *The function $B(a)$ has the following properties: (i) continuously differentiable, strictly increasing and strictly concave; (ii) $B(0) > \frac{u(0)}{1-\beta}$; and (iii) at least some of the total surplus from an insurance contract goes to the insurer.*

Part (i) of Assumption 2 lists technically desirable properties of the agent's outside option. Part (ii) ensures that the agent's inability to commit to a long-term contract (as analyzed in Sections 3 and 4) is relevant. Specifically, his outside option is such that the agent would eventually walk away from any contract that immiserates him. Part (iii) implies that the insurer's participation constraint—zero profits—will not bind. Here, we are thinking of environments where very rich agents can still only imperfectly insure against fluctuations in income and where principals have sufficient market power to extract some profits from an insurance agreement.

An example—autarky

One natural possibility for $B(a)$, which satisfies the assumed properties in Assumption 2, is to set it equal to the agent's value function in autarky $\Omega(a)$ determined by

$$\Omega(a) = \max_{\{a^i \geq 0\}_{i=1}^n} \sum_{i=1}^{n} \pi_i \left[u(ra + y^i - a^i) + \beta \Omega(a^i) \right]. \quad (1)$$

Suppose $u(0) > -\infty$ or $y^1 > 0$. Then the autarky value function $\Omega(a)$ satisfies all conditions in Assumption 2. By standard arguments (see Stokey, Lucas and Prescott, 1989), our assumptions on u ensure that $\Omega(a)$ satisfies property (i) in Assumption 2. Property (ii) is satisfied since $y^i > 0$ for all $i > 1$ (i.e., autarky expected consumption is positive at $a = 0$); property (iii) holds by construction.

The autarky (self-insurance) problem is a standard "income fluctuation problem", studied for instance in Schechtman and Escudero (1977) and Aiyagari (1994), among many others. Its solution features:

(i) imperfect consumption smoothing (c^i differs across states with different y^i);
(ii) consumption, c^i and asset choice, a^i for each income state increasing in current assets, a;
(iii) asset contraction (negative savings) in the lowest income state(s) and asset accumulation (positive savings) for some range of asset holdings in the highest income state(s).

Since $r < \beta^{-1}$ the agent only saves to insure against consumption volatility. Clearly then, an agent with more assets can do everything a poorer agent can, but is in a better position to self-insure against a long sequence of low outputs. The agent's inability to perfectly insure against income shocks, implies there is a demand for additional insurance. Specifically, the agent would be willing to sacrifice some consumption in order to increase smoothing across states of the world.

In section 4.3 we consider an alternative way to determine $B(a)$, by allowing the agent and principal to bargain over the terms of the insurance contract.

2.3. The first-best contract

We start by briefly characterizing the first-best contract in our setting, i.e., the optimal insurance contract when both parties can fully commit to an agreement signed at time zero. This is the natural benchmark against which we analyze the role of frictions—limited commitment and asset non-contractibility—in dynamic insurance. The first-best contract specifies state-contingent transfers/consumption and savings decisions. The only restriction in it is to satisfy ex-ante participation constraints. By our assumptions on $B(a)$ only the agent's participation constraint has to be imposed.

In the first-best, a long-term binding agreement is signed at the initial date specifying the complete path of history-contingent outcomes for all future periods and states. After that, the timeline is as follows. In the beginning of each period output is realized. Then, transfers from/to the agent take place. Finally, the agent consumes and saves the contracted amounts.

Note that due to the timing of events, if allowed to, the agent may, in principle, wish to deviate from the specified contract by varying his asset holdings and thus, his consumption. Below, we show that, with full commitment, this is not the case. That is, the first-best contract remains implementable even when agent's savings are non-contractible. Who controls asset accumulation does not matter. In contrast, in Section 4.1, we show that in absence of commitment, asset (non-)contractibility affects the resulting dynamic insurance contracts.

Proposition 1. *The first-best insurance contract has the following properties:*

(i) *for $r < R$, there is no asset accumulation by the agent, i.e., $a_t = 0$ for all $t > 0$; for $r = R$ setting $a_t = 0$ for all $t > 0$ is without loss of generality;*
(ii) *full insurance—agent's consumption is equalized across all output states in all periods, $c_t^i = c_t$ for all $i = 1, \ldots, n$ and $t \geq 0$.*
(iii) *weakly decreasing (strictly, if $\beta R < 1$) consumption profile, $c_t \geq c_{t+1}$, satisfying $u_c(c_t) = \beta R u_c(c_{t+1})$ at interior solution;*
(iv) *remains incentive-compatible when agent's savings are non-contractible—that is, if the agent were given the opportunity to vary his savings away from the level specified in the first-best contract, he would optimally choose not to.*

The results in parts *(ii)* and *(iii)* are standard. Since there is no private information or other incentive problems, the first-best contract offers the risk-averse agent equal consumption across output states in all periods. In addition, if the contracting parties discount at the same rate (the special case $R = \beta^{-1}$), the first-best consumption profile is constant over time, while if $R < \beta^{-1}$ agent's consumption is decreasing over time.

Regarding the role of asset accumulation by the agent in the first-best, Proposition 1*(i)* shows that, without loss of generality, agent's assets can be (must be, if $R > r$) extracted upfront and set to zero afterwards. In contrast, as we show in the next section, with lack of commitment, private asset accumulation by the agent is an integral part of insurance contracts. Proposition 1*(iv)* proves that asset contractibility does not affect the optimal contract: with full commitment the first-best can be implemented even when the principal has no control over the agent's (observable) assets since it remains incentive-compatible. The intuition is that the agent has no incentive to self-insure or protect himself against changes in the contract terms by saving privately since he faces full insurance in an infinitely-long contract.

Let $\bar{y} \equiv \sum_{i=1}^{n} \pi_i y^i > 0$ be expected output and $c_t \equiv ra_t + \tau_t - a_{t+1}$ be period consumption implied by a contract offering $\{\tau_t, a_{t+1}\}_{t=0}^{\infty}$. By Proposition 1 (see the proof in the Appendix for full details) we can write the first-best contracting problem as a two-stage problem. In the first stage, the insurer solves a static problem whereby he extracts the agent's initial assets a_0 and promises lifetime utility w from the next period on, subject to the agent's time-zero participation constraint:

$$\Pi^{FB}(a_0) = \max_{c_0, w} \; ra_0 + \bar{y} - c_0 + R^{-1}\tilde{\Pi}^{FB}(w) \tag{2}$$
$$\text{subject to} \;\; u(c_0) + \beta w - B(a_0) \geq 0,$$

where the function $\tilde{\Pi}^{FB}(w)$ solves the following, "second-stage" dynamic program of maximizing the insurer's profits subject to a promise-keeping constraint:

$$\tilde{\Pi}^{FB}(w) = \max_{c, w'} \; \bar{y} - c + R^{-1}\tilde{\Pi}^{FB}(w') \tag{3}$$
$$\text{subject to} \;\; u(c) + \beta w' - w = 0.$$

In the first-best, both the principal and the agent fully commit to follow the agreement signed at time zero. While the most efficient ex-ante, the first-best contract in our setting generically has the property that, ex-post, both parties would wish they could renege on it. For the agent, the reason to want to deviate is that his participation constraint is only imposed ex-ante. In the general case $R < \beta^{-1}$, the strictly decreasing consumption time path in the first-best contract implies that at some point in time the agent would be better-off by switching to his outside option with zero assets, $B(0)$. On the insurer's side, recall that when agent's initial assets a_0 are extracted at the start of the first-best contract, in compensation the agent is issued "credit" in the form of promised utility. Thus, even if the agent were committed to stay in the contract forever, as long as $a_0 > 0$, the insurer would like to renege on his past promises (worth $B(a_0)$ in total) and extract the most surplus from the current period onwards.[7]

3. Lack of Commitment

3.1. Markov-perfect contracts

Motivated by the various examples in the introduction, we relax the full commitment assumption on both the agent's and insurer's sides. Specifically, we assume that neither the principal nor the agent can bind themselves to a contract extending beyond the current period. That is, only one-period contracts can be enforced. Note that there are no penalties for failing to reach an agreement.

In our environment, individual punishment strategies such as, for example, the threat to never again sign a contract, are not credible—given our assumptions, at the beginning of each period there are always gains from insurance on the table. We adopt the solution concept of *Markov-perfect equilibrium* (Maskin and Tirole, 2001) and thus, characterize the best possible dynamic insurance contracts with (double-sided) lack of commitment, that are solely functions of fundamentals: beginning-of-period assets and current output realizations. In section 3.3 we show under what conditions the resulting equilibrium is equivalent to an infinitely-long contract in which the insurer can fully commit.

In a Markov-perfect equilibrium, the insurer offers the agent a contract for the current period, $\{\tau^i, a^i\}_{i=1}^n$ consisting of state-contingent transfers and end-of-period asset holdings, taking as given anticipated future interactions between himself and the agent. Specifically, given the agent's current assets a, the problem of the insurer today is to choose $\{\tau^i, a^i\}_{i=1}^n$ taking as given future decision rules $\{\mathcal{T}^i(a), \mathcal{A}^i(a)\}_{i=1}^n$ for transfers and assets, which induce profits $\Pi(a)$ and agent's continuation value $v(a)$.

[7] In the special case $R = \beta^{-1}$ the agent receives future discounted utility of $B(a_0)$ each period. After assets are extracted, the insurer would still like to renege on the promised utility $B(a_0)$ and drive the agent to his new outside option $B(0)$. Only in the knife-edge case, when both $R = \beta^{-1}$ *and* $a_0 = 0$ hold, does the first-best contract become a repeated static contract and as such, renegotiation-proof.

For now, let us maintain the assumption that agent's assets are contractible, i.e., the insurer can fully specify and control the agent's consumption and savings for the period (the case of non-contractible assets is analyzed in Section 4.1). In every period, the agent needs to decide whether to accept the currently offered contract or not. Given future interactions between agent and insurer which induce profits $\Pi(a)$ (to be defined below), a continuation value $v(a)$ and an outside option given by $B(a)$, the problem of the insurer facing an agent with assets a in the current period is

$$\max_{\{\tau^i, a^i \in \mathbb{A}\}_{i=1}^n} \sum_{i=1}^n \pi_i \left[y^i - \tau^i + R^{-1}\Pi(a^i) \right] \tag{4}$$

subject to the agent's participation constraint

$$\sum_{i=1}^n \pi_i \left[u(ra + \tau^i - a^i) + \beta v(a^i) \right] - B(a) \geq 0. \tag{5}$$

Note the difference between the time-zero participation constraint in the first stage of the first-best problem (2) and the period-by-period participation constraint (5) in the problem above. The following result follows immediately.

Lemma 1. *The participation constraint* (5) *binds and* $v(a) = B(a)$ *for all* $a \in \mathbb{A}$.

Intuitively, today's principal leaves no surplus to the agent regardless of how the surplus is distributed in the future. In particular, if for some reason the insurer tomorrow left the agent better-off than his outside option, because of lack of commitment, the insurer today would still find it optimal to extract all the surplus. As a result, the agent is always kept exactly as well-off as at his outside option, for all asset levels. Thus, the participation constraint in the problem above simplifies to

$$\sum_{i=1}^n \pi_i [u(ra + \tau^i - a^i) + \beta B(a^i)] - B(a) \geq 0. \tag{6}$$

The assumed strict concavity of u implies that full insurance is optimal, i.e., $c^i \equiv ra + \tau^i - a^i$ is equalized across states $i = 1, \ldots, n$ (formally, this can be seen by taking the first-order conditions with respect to τ^i). Intuitively, in the absence of incentive-provision concerns it is inefficient to have the risk-averse agent bear any risk across output-states. What is key for this result is that the parties cannot renege after output is realized. Note again the contrast with the "limited commitment" literature on optimal insurance (e.g., Kocherlakota, 1996; Ligon, Thomas and Worrall, 2002; Krueger and Uhlig, 2006) where agents can renege on the insurance scheme *after* observing the output and as a result only partial insurance may be sustainable. In addition, the first-order conditions with respect to a^i of problem (4) subject to (6) are fully symmetric across output-states, so assuming a symmetric solution we must have $a^i = a^j = a'$ and $\tau^i = \tau^j = \tau$ for all $i, j = 1, \ldots, n$.

Using these results, we formally define a *Markov-perfect equilibrium* and a *Markov-perfect contract* in our setting as follows.

Definition 1. *(i) A* ***Markov-perfect equilibrium (MPE)*** *is a set of functions* $\{\mathcal{T}, \mathcal{A}, \Pi\} : \mathbb{A} \to \mathbb{R} \times \mathbb{A} \times \mathbb{R}_+$ *defined such that, for all* $a \in \mathbb{A}$*:*

$$\{\mathcal{T}(a), \mathcal{A}(a)\} = \underset{\tau, a' \in \mathbb{A}}{\operatorname{argmax}} \ \bar{y} - \tau + R^{-1}\Pi(a')$$

subject to

$$u(ra + \tau - a') + \beta B(a') - B(a) \geq 0;$$

and

$$\Pi(a) = \bar{y} - \mathcal{T}(a) + R^{-1}\Pi(\mathcal{A}(a)).$$

(ii) For any $a \in \mathbb{A}$*, the* ***Markov-perfect contract*** *implied by a MPE, as defined in (i), is the transfer and savings pair:* $\{\tau = \mathcal{T}(a),\, a' = \mathcal{A}(a)\}$.

To simplify the exposition, denote by $\mathcal{C}(a) \equiv ra + \mathcal{T}(a) - \mathcal{A}(a)$ the consumption function in an MPE and let $c \equiv ra + \tau - a'$ denote the consumption level for any contract $\{\tau, a'\}$ given any $a \in \mathbb{A}$. Using Definition 1, re-write the Markov-perfect contracting problem of (4) subject to (6) in a mathematically equivalent way as:

$$\Pi(a) = \max_{c,a' \geq 0} \bar{y} + ra - c - a' + R^{-1}\Pi(a') \tag{7}$$

subject to

$$u(c) + \beta B(a') - B(a) = 0. \tag{8}$$

The insurer takes the outside option function $B(a)$ as exogenously given—he cannot control its value except via agent's assets; e.g., as in the case of autarky when $B(a) = \Omega(a)$ for all $a \in \mathbb{A}$. We study an example of "endogenous" outside option in Section 4.3. To solve problem (7) we need to find a fixed-point in the function Π. Our assumptions on $B(a)$ imply $B(a) \geq B(0) > u(0) + \beta B(0)$. Thus, the constraint set is non-empty for all $a \in \mathbb{A}$. Since the set $\mathbb{A}$ is compact, standard contraction mapping arguments ensure the existence and continuity of the value function Π (Stokey, Lucas and Prescott, hereafter, SLP, 1989; see also Krueger and Uhlig, 2006). More specifically, similarly to Krueger and Uhlig (2006), using that u and B are strictly increasing and changing variables to $z \equiv B(a)$, we can rewrite problem (7) as

$$\Pi^*(z) = \max_{z'} \bar{y} + F(z, z') + R^{-1}\Pi^*(z'), \tag{9}$$

where $z, z' \in [B(0), B(\bar{a})]$ and $F(z, z') \equiv rB^{-1}(z) - u^{-1}(z - \beta z') - B^{-1}(z')$. It is easy to verify that SLP Assumptions 4.3, 4.4 and 4.8 are satisfied. By SLP Theorem 4.6 this implies uniqueness of the fixed point Π. A sufficient condition for continuous single-valued policy, z' in problem (9), and hence continuous functions $\mathcal{T}$, $\mathcal{C}$ and $\mathcal{A}$ in the original problem as given in Definition 1, is $F(z, z')$ to be strictly concave (see SLP, 1989, Assumption 4.7 and Theorem 4.8). Assume u and B are such that this holds. Assuming strict concavity of F thus implies existence and uniqueness of MPE as given in Definition 1. Next, by the differentiability of u and B, F is continuously differentiable on the interior of its domain. This implies that Π^* is differentiable on the interior of its domain and and hence Π is differentiable on the interior of $\mathbb{A}$ (see SLP, 1989, Assumption 4.9 and Theorem 4.11). Figure 1 illustrates a numerical example of the value and policy functions satisfying these properties.

The first-order conditions of problem (7)—(8) with respect to c and a' are

$$-1 + \lambda u_c(c) = 0$$
$$-1 + R^{-1}\Pi_a(a') + \lambda\beta B_a(a') + \zeta = 0,$$

where λ and ζ are the Lagrange multipliers associated with the participation constraint and the non-negativity constraint on future assets, respectively.

Note that $\lambda = 1/u_c > 0$. The envelope condition implies

$$\Pi_a(a) = r - \frac{B_a(a)}{u_c(c)}. \tag{10}$$

Plugging (10) and $\lambda = 1/u_c$ into the first-order condition for a', the equations characterizing a Markov-perfect insurance contract for any $a \in \mathbb{A}$ are the participation constraint (8) and

$$r - R + B_a(a') \left[\frac{\beta R}{u_c(c)} - \frac{1}{u_c(c')} \right] + \zeta R = 0. \tag{11}$$

Proposition 2. *A Markov-perfect equilibrium has the following properties:*

- (i) *(weakly) decreasing asset profile over time:* $\exists \hat{a} \in (0, \infty)$ *such that* $\mathcal{A}(a) = 0$ *for all* $a \in [0, \hat{a}] \cap \mathbb{A}$ *and* $0 < \mathcal{A}(a) < a$ *for all* $a > \hat{a}$*;*
- (ii) *decreasing consumption profile over time,* $\mathcal{C}(\mathcal{A}(a)) < \mathcal{C}(a)$ *for all* $a > 0$*;*
- (iii) *consumption,* $\mathcal{C}(a)$ *and savings,* $\mathcal{A}(a)$ *(weakly) increasing in current assets,* a*;*
- (iv) *zero assets in finite time and positive long-run consumption: there exists* $T < \infty$ *such that* $a_T \equiv \mathcal{A}(\mathcal{A}(\ldots\mathcal{A}(a_0)\ldots))(T - \text{times}) = 0$*, for all* $a_0 \in \mathbb{A}$*, and* $\mathcal{C}(0) > 0$*.*

We assume that $\bar{a}$ is chosen such that $\hat{a} < \bar{a}$. Figure 1 displays a computed example of a Markov-perfect equilibrium, which illustrates the results in Proposition 2. As a reference, the MPE is compared to autarky, as defined in (1). Displayed are transfers, savings, consumption and net present value profits in a MPE, all as functions of assets. As we can see, in the long-run, assets are depleted and consumption converges to a positive amount. In this particular example, long-run MPE consumption is slightly above autarky consumption in the high-output state for an agent holding zero assets.

Figure 1: Markov-perfect equilibrium

[Figure 1 about here]

Note: Solid lines correspond to variables in a MPE; dashed-lines correspond to variables in autarky. There are two output states, labeled y^L and y^H. Asset accumulation and consumption in autarky are given by: a^H, a^L, c^H and c^L. The assumed parameterization is: $u(c) = \ln c$, $\beta = 0.93$; $r = 1.06$; $R = 1.07$; $y^1 = 0.1$, $y^2 = 0.3$ and $\pi^1 = 0.5$.

3.2. Comparison with the first-best

Compare the results of Proposition 2 characterizing Markov-perfect insurance contracts with the properties of the first-best contract in Proposition 1. First, the role of asset accumulation is very different. Intuitively, while in the first-best it is (weakly) optimal to extract all assets at time zero, in a MPE, without commitment by the insurer any positive initial wealth can only be gradually reduced over time as dictated by the per-period participation constraints.

Second, the insurer's lack of commitment distorts the slope of the consumption profile. In the first-best (see Proposition 1), we have the standard Euler equation, $u_c(c) = \beta R u_c(c')$. In contrast, with lack of commitment we can rewrite (11) at an interior solution (i.e., for $a > \hat{a}$) as the following modified Euler equation:

$$u_c(c) = \beta R u_c(c') \left[1 - \frac{(R-r)}{R} \frac{u_c(c)}{\beta B_a(a')}\right]. \qquad (12)$$

The non-positive "wedge", $-\frac{(R-r)}{R}\frac{u_c(c)}{\beta B_a(a')}$ reflects the inefficiency introduced by the principal's lack of commitment. In the first-best contract, the insurer extracts all the agent's assets and invests them in the more productive asset accumulation technology with return R (this is without loss of generality when $r = R$). When the principal lacks commitment, he cannot compensate the agent for such immediate asset extraction with promised utility (or anything else), and thus he can only run down the agent's savings gradually, respecting the participation constraint. This distorts efficiency. In the resulting Euler equation wedge, the term $\frac{u_c(c)}{\beta B_a(a')}$ states the trade-off faced by the principal: each additional unit of assets he would like to extract lowers the agent's future outside option by βB_a and hence, needs to be compensated with additional consumption now, worth u_c. Naturally, the size of the distortion arising from not being able to extract assets optimally is proportional to the difference between the insurer's and the agent's intertemporal rates of return, $R - r$.

When $r < R$, (12) implies $u_c(c) < \beta R u_c(c')$, i.e., the agent is forced to consume more today compared to the first-best, although the time-profile is steeper. In the special case $r = R < \beta^{-1}$ consumption in the first-best and Markov settings is reduced at the same rate, but in the first-best it goes down further (converges to zero rather than to a positive value—see below).

Third, the agent's inability to commit to remain in the contract beyond the current period, causes Markov-perfect contracts to differ from the first-best in terms of long-run consumption. In the first-best contract, consumption converges to zero when $R < \beta^{-1}$ and is constant over time, delivering present value utility of $B(a_0)$, when $R = \beta^{-1}$. In contrast, with lack of commitment, consumption decreases over time and converges to the strictly positive value delivering present value utility $B(0)$. The reason is that the agent can always walk away. Thus, even though full insurance is provided, once assets are depleted, the principal can only retain the agent by offering him $B(0)$ from then on, which implies positive long-run consumption since by assumption, $B(0) > u(0)/(1-\beta)$.[8] Clearly, the first-best and no-commitment contracts coincide only in the special case $R = \beta^{-1}$ *and* $a_0 = 0$ (see also footnote 7).

3.3. Markov-perfect insurance vs. one-sided commitment

To further understand the role of commitment by each contract side in the dynamic insurance problem, we compare and contrast our Markov-perfect contracts defined and characterized above to the situation in which only the insurer has full long-term commitment ability. More precisely, consider a "one-sided commitment" contract where, like in the first-best (Section 2.2), the insurer is able to commit to an infinite sequence of state-contingent transfers at time zero while, as in Section 3.1 the agent is allowed to walk away from the contract at the beginning of each period before output is realized and cannot renege on the contract for the period if he decides to stay on. This implies that, like in a Markov-perfect contract, the agent's participation constraint must be satisfied in every period.

By the same arguments as in Section 2.3, full insurance obtains. Also, as in Proposition 1, without loss of generality, the principal extracts all assets in the initial period. Thus, as we did for the first-best in (2)–(3) in Section 2.2, we can write the insurer's problem with one-sided commitment as a two-stage problem. In the first stage, the insurer solves a static problem whereby he extracts the agent's initial assets, a_0, and promises in exchange lifetime utility w from next period onwards, subject to the constraint $w \geq B(0)$,

$$\Pi^{OS}(a_0) = \max_{c_0, w} \ \bar{y} + ra_0 - c_0 + R^{-1}\tilde{\Pi}^{OS}(w) \tag{13}$$

subject to

$$\begin{aligned} u(c_0) + \beta w - B(a_0) &= 0 \\ w - B(0) &\geq 0. \end{aligned}$$

The function $\tilde{\Pi}^{OS}(w)$ is the solution to the following "second-stage" dynamic program for any $w \geq B(0)$:

$$\tilde{\Pi}^{OS}(w) = \max_{c, w'} \ \bar{y} - c + R^{-1}\tilde{\Pi}^{OS}(w') \tag{14}$$

subject to

$$\begin{aligned} u(c) + \beta w' - w &= 0 \\ w' - B(0) &\geq 0. \end{aligned}$$

The key differences with the first-best contract solving (2)—(3) are the additional inequality constraints $w - B(0) \geq 0$ and $w' - B(0) \geq 0$ on future promised utility at each stage. These constraints embody the agent's inability to commit not to renege on the contract at the beginning of any period and obtain his outside continuation utility $B(0)$. In the first-best, promised utility can fall below $B(0)$ in the long-run (e.g., in the case $\beta R < 1$ when consumption converges to zero).

Proposition 3. ***Markov-perfect vs. one-sided commitment contracts.***

(i) *If $r = R$ or $a_0 \in [0, \hat{a}]$, where $\hat{a}$ is defined in Proposition 2(i), then, from the same initial asset level, a Markov-perfect equilibrium where the insurer solves problem* (7)—(8) *implies identical consumption sequence $\{c_t\}_{t=0}^{\infty}$ and time-zero discounted profits as a one-sided commitment contract defined by* (13)—(14).

(ii) *If $r < R$ and $a_0 > \hat{a}$ where $\hat{a}$ is defined in Proposition 2(i), then, from the same initial assets level, insurance contracts in a Markov-perfect equilibrium differ from one-sided commitment contracts in their implied consumption sequence $\{c_t\}_{t=0}^{\infty}$ and profits. Specifically:*

  (a) *time-zero discounted profits are strictly higher with one-sided commitment;*

[8] Note that this result does not rely on the assumed borrowing constraint. Even if the agent were allowed to borrow up to the natural borrowing limit (a case we subsume if $y^1 = 0$), his outside option $B(0)$ dominates the present value of zero consumption forever.

(b) *agent's consumption* c_t *is reduced faster over time in a Markov-perfect equilibrium, but as long as* $R\beta < 1$*, long-run consumption is the same in both settings.*

The results in Proposition 3 expand on Section 3.2 to further clarify the role of commitment by the insurer. Part *(i)* establishes that, when the insurer and agent face the same rate of return, our Markov-perfect insurance contracts from Section 3.1 are equivalent to optimal dynamic insurance contracts in which the insurer can commit to a long-term ex-ante contract but the agent cannot. Intuitively, when both parties can carry assets at the same rate of return, promised utility, w' and the contractible assets, a' are fully interchangeable from the point of view of the insurer—everything that can be implemented with one of these two instruments can be exactly replicated with the other, because of the binding participation constraint each period.

Proposition 3, part *(ii)* shows that one-sided commitment and Markov-perfect contracts do differ, however, when $r < R$ and $a_0 > \hat{a}$ (the interior solution case). One-sided commitment contracts generate more surplus than Markov-perfect contracts, starting from the same initial asset level. Intuitively, the principal's long-term commitment ability in the one-sided setting enables him to extract the agent's assets in the initial period in exchange of promised utility and use his superior rate of return to raise total surplus. Obviously, for any gains to be had from this, the agent's assets must be positive in a MPE. The agent agrees to effectively 'hand over' his assets at $t = 0$ because he knows that the principal cannot renege on his promise to compensate him with future transfers. The rate of return differential causes consumption to be reduced faster in an MPE but, as long as $R\beta < 1$, both settings yield the same positive long-run consumption level determined by the agent's participation constraint evaluated at current and future assets equal to zero. This is in contrast with the first-best (see Proposition 1) where consumption converges to zero in the long run. The reason is that the agent can always go to his outside option if offered too low consumption.

4. Extensions

4.1. Non-contractible assets

The results in Sections 2.3, 3.1 and 3.3 show that when the insurer lacks ability to commit to a long-term contract, there are non-trivial equilibrium asset dynamics. Given that, in general, the agent holds assets in a Markov-perfect equilibrium, a natural question is whether the best dynamic insurance contract would differ if the insurer were unable to control agent's (observable) savings. This is important, as one can imagine many situations in which private or public insurance providers can observe or infer agents' asset positions but cannot force agents to hold particular asset levels or, equivalently, to control agents' consumption directly. Below we show that which contractual side controls the savings choice does affect MPE's nature as long as the insurer obtains a positive fraction of joint surplus (see Section 4.2 for the remaining case of perfectly-competitive insurers).

If agent's assets are observable but non-contractible, the insurer must induce the desired levels of c and a' through a suitable choice of transfers. Consider the problem of an agent with assets a who signs an insurance contract that specifies transfers τ^i, $i = 1, \dots, n$:

$$\max_{\{a^i \geq 0\}_{i=1}^n} \sum_{i=1}^n \pi_i \big[u(ra + \tau^i - a^i) + \beta B(a^i)\big],$$

where again we use Lemma 5 to specify the continuation value. The first-order condition of the agent's problem with respect to savings, a^i, $i = 1, \dots, n$ is

$$-u_c(c^i) + \beta B_a(a^i) + \xi^i = 0, \tag{15}$$

where ξ^i denotes the multiplier on the non-negativity constraint for a^i. We will use the agent's first-order conditions in a^i (15) as incentive-compatibility constraints for the insurer with respect to the agent's choice of a^i. This "first-order approach" is valid here, since by our assumptions on u and B the agent's problem is concave in a^i. The insurer's problem when the agent's assets are non-contractible is thus analogous to the problem with contractible assets from Section 3.1, (7)—(8), but with the additional constraints,

$$u_c(c^i) - \beta B_a(a^i) \geq 0 \text{ for } i = 1, \dots, n. \tag{16}$$

As in Section 3.1, it is easy to verify that the first-order conditions imply full insurance. A *Markov-perfect equilibrium with non-contractible assets*, $\{\mathcal{T}^N, \mathcal{A}^N, \Pi^N\}$ is then defined analogously to Definition 1, but with the additional incentive-compatibility constraints, (16) included in the insurer's problem. A *Markov-perfect contract with non-contractible assets* for any given $a \in \mathbb{A}$ is simply the offered equilibrium transfer $\{\tau = \mathcal{T}^N(a)\}$.

Although agent's savings are no longer contractible, consumption and current assets are still observable. Therefore, as in Section 3.1 we can rewrite the problem of the insurer as if choosing consumption, c (rather than transfers) and next period's asset a' subject to the participation constraint and the incentive-compatibility constraint for a',

$$\Pi^N(a) = \max_{c,a' \geq 0} \bar{y} + ra - c - a' + R^{-1}\Pi^N(a') \tag{17}$$

subject to

$$u(c) + \beta B(a') - B(a) = 0 \tag{18}$$

$$u_c(c) - \beta B_a(a') \geq 0, \text{ with equality if } a' > 0. \tag{19}$$

The next Lemma characterizes some properties of Markov-perfect contracts with non-contractible assets which solve (17)—(19).

Lemma 2. *Savings, $\mathcal{A}^N(a)$ and consumption, $\mathcal{C}^N(a)$ in a Markov-perfect equilibrium with non-contractible assets are (weakly) increasing in a.*

Let $a'^N = \mathcal{A}^N(a)$ and $c^N = \mathcal{C}^N(a)$ be the optimal asset and consumption choices in the non-contractible problem. Notice that it cannot be true that $a'^N = 0$ for all $a \in \mathbb{A}$. Suppose this were true. We would then have from (19) that $u_c(c^N) - \beta B_a(0) \geq 0$ and from (18) that $u(c^N) + \beta B(0) = B(a)$ for all a. Since $B(a)$ is strictly increasing in a, the latter implies that c^N is strictly increasing in a so the left-hand side of the inequality $u_c(c^N) - \beta B_a(0) \geq 0$ is strictly decreasing in a. But then it must be that $u_c(c^N) - \beta B_a(0) > 0$ for all $a \in [0, \bar{a})$—i.e., constraint (19) is not binding for those a (its Lagrange multiplier is zero). Because of this, for any such a the solution to problem (17) subject to (18) and (19) must be the same as the solution to the same problem with constraint (19) removed. However, this latter problem is exactly the Markov-perfect insurance problem with contractible assets, (7) subject to (8). The solution equivalence implies $\mathcal{A}(a) = \mathcal{A}^N(a) = 0$ for all $a \in [0, \bar{a})$, which contradicts Proposition 2 where we showed that $\mathcal{A}(a) > 0$ for any $\bar{a} > a > \hat{a}$. Thus, it cannot be that $a'^N = 0$ for all $a \in \mathbb{A}$ and so there exists some $a^* > 0$ for which both $a' = \mathcal{A}(a) > 0$ and $a'^N = \mathcal{A}^N(a) > 0$ for all $a \in [a^*, \bar{a})$. This result is used in the next proposition which characterizes the role of asset contractibility in Markov-perfect insurance contracts.

Proposition 4. ***The role of asset contractibility in Markov-perfect contracts.***

(i) *MPE with contractible assets $\{\mathcal{A}, \mathcal{C}, \Pi\}$ differ from MPE with non-contractible assets, $\{\mathcal{A}^N, \mathcal{C}^N, \Pi^N\}$. That is, there exist one or more $a \in \mathbb{A}$ for which $\{\mathcal{A}(a), \mathcal{C}(a), \Pi(a)\} \neq \{\mathcal{A}^N(a), \mathcal{C}^N(a), \Pi^N(a)\}$—it matters who controls asset accumulation.*

(ii) *Specifically, take any asset level a for which $a' > 0$ and $a'^N > 0$. For any such a, if $\Pi_a(a') < 0$ then savings are strictly higher and consumption lower in the non-contractible assets case, i.e., $a' < a'^N$ and $c > c^N$. The opposite is true if $\Pi_a(a') > 0$. It is not possible that $\Pi_a(a') = 0$ for all $a > \hat{a}$.*

Recall from Proposition 1 that the first-best contract remains incentive compatible when savings are non-contractible since the agent has no incentive to save given the full insurance provided and the fact that any side savings do not alter the terms of the contract set at time zero. This is no longer the case here. The lack of commitment friction misaligns the asset accumulation incentives of the contracting parties. Intuitively, the agent can use his ability to save privately to change his outside option and counter the principal's driving him to the bound $B(0)$. The agent wants to raise his outside option and ensure higher future transfers when able to control his assets.

Proposition 4 shows the existence of one or more asset levels for which the contracts with and without asset contractibility differ. It does not rule out the possibility that these two contracts coincide for some asset levels. Theoretically, this could occur—for example, if Markov-perfect contracts with non-contractible assets feature a binding non-negativity constraint for a close to zero.

To further characterize Markov-perfect insurance contracts with non-contractible assets, assume that $B(a)$ is twice continuously differentiable and focus on the case of interior solution for assets, $a'^N > 0$, where $u_c(c) - \beta B_a(a') = 0$. With λ and μ the Lagrange multipliers associated with (18) and (19), respectively, the first-order conditions of problem (17) are

$$-1 + \lambda u_c(c) + \mu u_{cc}(c) = 0 \tag{20}$$

$$-1 + R^{-1}\Pi_a^N(a') + \lambda\beta B_a(a') - \mu\beta B_{aa}(a') = 0. \tag{21}$$

Using $u_c(c) = \beta B_a(a')$ from (19) and (20), equation (21) can be rewritten as

$$R^{-1}\Pi_a^N(a') - \mu[u_{cc}(c) + \beta B_{aa}(a')] = 0,$$

which, by the concavity of u and B and since $\mu \geq 0$ at interior solution, implies $\Pi_a^N(a') \leq 0$. This yields the following result.

Lemma 3. *The insurer's profits in the non-contractible assets case, Π^N are decreasing in agent's assets at any asset level $\tilde{a} > 0$ belonging to the range of $\mathcal{A}^N(a)$ for $a \in \mathbb{A}$.*

Conceptually, the insurer's profits could go up in agent's assets simply since there is more wealth to extract—note the term ra in the objective (17). On the other hand, richer agents who can control their private savings demand less additional insurance, which reduces the insurer's profits. Lemma 3 shows that the latter effect dominates the former.

The system of equations formed by the constraints to the insurer's problem with non-contractible assets and its first-order conditions, (18)—(21) permits a steady-state with positive assets. In such a case, Markov-perfect contracts with and without asset contractibility differ for all $a \in A$ and imply different long-run allocations. Figure 2 shows a numerical example illustrating this point using the same parameterization as Figure 1. As we can see, long-run savings are zero at some positive asset level for the case of non-contractible assets. We also display profits, which are naturally lower for the non-contractible case. We omit displaying transfers and consumption functions since they are visually indistinguishable.

Figure 2: Effect of asset contractibility in MPE

[Figure 2 about here]

Note: Solid lines correspond to MPE with contractible assets; dashed lines correspond to MPE with non-contractible assets. The assumed parameterization is: $u(c) = \ln c$, $\beta = 0.93$; $r = 1.06$; $R = 1.07$; $y^1 = 0.1$, $y^2 = 0.3$ and $\pi^1 = 0.5$.

4.2. Perfectly competitive insurers

In this section we characterize the case of perfectly competitive insurance market with free entry by insurers. Free entry implies that optimal contracts maximize the agent's expected present value utility subject to a zero-profits constraint for the insurer. More specifically, perfect competition in the insurance market results in zero per-period profits for any sub-market indexed by agent's assets holdings, a. Cross-subsidization across different asset levels or time periods is ruled out by the possibility of free entry each period.

Define a Markov-perfect equilibrium and Markov-perfect contracts analogously to Definition 1 (details omitted to save space). Denoting $c^i \equiv ra + \tau^i - a^i$, the contracting problem of a perfectly competitive insurer with double-sided lack of commitment and contractible assets can be written recursively as (see also Karaivanov and Martin, 2011 for detailed discussion of the free-entry case in a more general insurance setting with moral hazard):

$$v(a) = \max_{\{\tau^i, a^i \geq 0\}_{i=1}^n} \sum_{i=1}^n \pi_i \left[u(c^i) + \beta v(a^i)\right]$$

subject to

$$\sum_{i=1}^{n} \pi_i \left[y^i - \tau^i \right] = 0.$$

There is no need for a participation constraint for the agent as he obtains all surplus from the insurance relationship. It is easy to show using the first-order conditions that full insurance obtains once again, i.e., $c^i = c$, $\tau^i = \tau$ and $a^i = a'$ for all $i = 1, \ldots, n$. Hence, from the zero-profits condition, $\tau = \bar{y}$ and $c = \bar{y} + ra - a'$ which simplifies the above contracting problem to:

$$v(a) = \max_{a' \geq 0} \; u(\bar{y} + ra - a') + \beta v(a'). \tag{22}$$

Under our assumptions on u and $\beta r < 1$, Problem (22) is equivalent to Example 5.17 in Stokey, Lucas and Prescott (1989, pp. 126-28). Thus, the value function v is continuous, strictly increasing, strictly concave and continuously differentiable for any $a \in \mathbb{A}$. The policy function, $a' = \mathcal{A}(a)$ is unique, continuous and weakly increasing.

Proposition 5. *In Markov-perfect equilibrium with free entry by insurers:*

(i) *at an interior solution for a', the standard Euler equation applies,*

$$u_c(c) = \beta r u_c(c'); \tag{23}$$

(ii) *properties (i)—(iv) listed in Proposition 2 hold;*

(iii) *it does not matter whether agent's assets are contractible or not, that is, Markov-perfect contracts with contractible and non-contractible assets coincide.*

Contrast the optimality condition, (23) with its counterpart, (12) from the non-competitive case analyzed earlier. First, the free entry and lack of commitment imply that no assets can be carried over time by the insurer at the rate of return R—only the agent's return r matters. Second, the free-entry assumption implies that there is no conflict between the insurer and agent about the optimal path of consumption. The optimality condition (23) shows the prescribed consumption path follows a standard Euler equation. But this is precisely what the agent would do if he controlled his own savings. However, unless $r = R$, the parties' inability to commit still matters for the consumption path—the Euler equation in (23) differs from the one in the first-best, see Proposition 1(iii). Third, note the contrast between Proposition 5(iii) and Proposition 4 from the non-competitive case. Intuitively, a perfectly competitive insurer earning zero profits has no incentive to manipulate the agent's future value by imposing a different time path for assets as an insurer with some market power would.

4.3. Bargaining

So far, we have considered two extreme ways of dividing the surplus in insurance contracts. First, in Section 3, we solved the problem of how much profit the principal can extract for a given agent outside option $B(a)$. Then, in Section 4.2, we solved the case of perfect competition among insurers, when all the surplus of an insurance contract goes to the agent. In this section, we characterize a Markov-perfect dynamic insurance contract which results when insurer and agent bargain over the terms of the agreement every period. We adopt the proportional solution of Kalai (1977) and assume that the outside option for the agent is autarky $\Omega(a)$, as defined in Section 2, and the outside option for the principal is zero profits.

Let θ be the agent's bargaining weight, which in Kalai's proportional solution equals the agent's share of the total surplus. Focus on the case $\theta \in (0, 1)$ so that the participation constraints of the contracting parties,

$$u(c) + \beta v(a') - \Omega(a) \geq 0$$
$$\bar{y} + ra - c - a' + R^{-1}\Pi(a') \geq 0,$$

do not bind. As in previous sections, we use $c \equiv ra + \tau - a'$, to replace transfers with consumption and assets in all expressions.

The insurance contract with bargaining is obtained from the proportional solution (Kalai, 1977) which satisfies

$$\max_{c,a'\in\mathbb{A}} \; \bar{y} + ra - c - a' + R^{-1}\Pi(a') \tag{24}$$

subject to

$$(1-\theta)\left[u(c) + \beta v(a') - \Omega(a)\right] - \theta\left[\bar{y} + ra - c - a' + R^{-1}\Pi(a')\right] = 0. \tag{25}$$

A Markov-perfect equilibrium with bargaining over the terms of the insurance contract is defined, in a way analogous to Definition 1, by a set of functions $\{\mathcal{C}, \mathcal{A}, v, \Pi\}$ that for all $a \in \mathbb{A}$ solve problem (24)—(25) and where

$$v(a) = u(\mathcal{C}(a)) + \beta v(\mathcal{A}(a)) \tag{26}$$

$$\Pi(a) = \bar{y} + ra - \mathcal{C}(a) - \mathcal{A}(a) + R^{-1}\Pi(\mathcal{A}(a)). \tag{27}$$

The resulting equilibrium value function $v(a)$ for the agent is an example of how to endogenize the agent's outside option from the previous sections, $B(a)$.

At an interior solution, the first-order conditions of problem (24)—(25) are

$$-1 + \lambda\left[(1-\theta)u_c(c) + \theta\right] = 0$$

$$-1 + R^{-1}\Pi_a(a') + \lambda\left[(1-\theta)\beta v_a(a') - \theta(-1 + R^{-1}\Pi_a(a'))\right] = 0,$$

where λ is the Lagrange multiplier associated with the surplus-splitting constraint, (25). From the first equation above, $\lambda = [(1-\theta)u_c(c) + \theta]^{-1}$ and thus, plugging in for λ, the second equation becomes

$$-u_c(c) + \beta v_a(a') + R^{-1}u_c(c)\Pi_a(a') = 0. \tag{28}$$

The problem defined by (24)—(25) determines the insurer's equilibrium profits as a function of assets, $\Pi(a)$. Thus, by the Envelope theorem

$$\Pi_a(a) = r - \lambda[(1-\theta)\Omega_a(a) + \theta r],$$

which, plugging in for λ from above, implies

$$\Pi_a(a) = \frac{ru_c(c) - \Omega_a(a)}{u_c(c) + \frac{\theta}{1-\theta}}. \tag{29}$$

Using (26)—(27), the proportional surplus-splitting rule (25) can be written as $(1-\theta)[v(a) - \Omega(a)] - \theta\Pi(a) = 0$ for all $a \in \mathbb{A}$. Totally differentiating this expression with respect to a, and rearranging implies

$$v_a(a) = \frac{\theta}{1-\theta}\,\Pi_a(a) + \Omega_a(a).$$

Using (29), we obtain

$$v_a(a) = \frac{u_c(c)[\frac{r\theta}{1-\theta} + \Omega_a(a)]}{u_c(c) + \frac{\theta}{1-\theta}}. \tag{30}$$

Plugging in for (29) and (30) evaluated one period ahead into (28) and rearranging, we obtain the following Euler equation:

$$u_c(c) = \beta R u_c(c')\left[\frac{\theta r + (1-\theta)[\Omega_a(a') - u_c(c)(R-r)(\beta R)^{-1}]}{\theta R + (1-\theta)\Omega_a(a')}\right]. \tag{31}$$

Note that as the bargaining power of the agent $\theta \to 0$ the Euler equation (31) converges to

$$u_c(c) = \beta R u_c(c')\left[1 - \frac{(R-r)}{R}\frac{u_c(c)}{\beta\Omega_a(a')}\right].$$

If we set $B(a) = \Omega(a)$ in Section 3, then the Euler equation (12), which characterizes an interior solution of the contracting problem with lack of commitment and a monopolist insurer, coincides with the expression above. In other words, if the contracting parties lack commitment and autarky is the agent's outside option, then as we decrease the agent's bargaining power we converge to the contract with a monopolist insurer.[9]

On the other hand, as the bargaining power of the agent $\theta \to 1$, expression (31) converges to $u_c(c) = \beta r u_c(c')$, which is the same as (23), the Euler equation under perfect competition among insurers. In this case, as in Section 4.2, the insurance contract does not depend on the agent's outside option.

5. Concluding Remarks

We study dynamic insurance problems in which the contracting parties cannot commit to long-term agreements, but are able to carry resources over time, at potentially different rates of return. We find that the gains from insurer's commitment to infinitely-long contracts, as opposed to one-period contracts, come from exploiting a superior rate of return on carrying assets over time. Perhaps surprisingly, there are no other efficiency gains from long-term commitment by the insurer in our setting. If the agent holds sufficiently low initial assets or if the insurer and agent share the same rate of return, one-period Markov-perfect insurance contracts generate equivalent consumption paths and welfare as those arising when the insurer can commit to an infinitely-long contract subject to per-period participation constraints by the agent.

In contrast, an agent's ability to commit long-term could be exploited by an insurer to generate higher ex-ante surplus by driving agent's consumption towards its lowest bound. This is impossible if the agent is free to leave the contract each period to an outside option with present value exceeding the value of zero consumption forever. Thus, lack of long-term commitment by the agent always introduces inefficiency relative to the first-best if the optimal consumption profile is decreasing in time (the case $\beta R < 1$ in our setting). The insurer's inability to commit and hence, the need to carry assets over time at an inferior rate of return (r) implies that asset accumulation and contractibility play a key role in Markov-perfect insurance contracts, yielding non-trivial asset dynamics. This is in contrast to the trivial role of observable assets in contracts with commitment.

The time-profiles of agent's consumption and insurer's profits depend critically on the parties' ability to commit and on the degree of the insurer's market power. If the insurer can commit to a long-term contract, production efficiency and total surplus are maximized since agent's assets can be immediately invested in the superior return technology in exchange for promises of future consumption. In contrast, if insurers lack commitment and there is free entry, no assets can be carried at the superior rate of return in a Markov-perfect equilibrium—instead, the agent's initial assets are invested in the low-return technology until they are depleted. When the insurer has market power, the lack of commitment limits his ability to efficiently extract agent's assets (they are driven to zero only gradually) leading to distorted consumption and profit profiles and lower total surplus.

Our approach and findings can be related to the literature on incomplete contracts and political economy.[10] For example, Battaglini and Palfrey (2012) study a dynamic problem of surplus splitting in a voting game. Each period one of three agents proposes a sharing rule which is accepted if there is majority in favor or, if rejected, the last-period's sharing rule is applied. Therefore, as in our analysis, a policy chosen today affects next period's status quo and determines tomorrow's endogenous outside option. Battaglini and Palfrey also focus on Markov strategies and equilibria dependent only on payoff-relevant events. Experimentally, they find evidence for concave utility and show that agents' discount factors affect in a significant way the resulting equilibria.

In a companion paper (Karaivanov and Martin, 2011) we keep the double-sided lack of commitment assumption and extend some of the results presented here to a setting in which we endogenize the agent's income process and introduce a moral hazard problem due to unobserved effort. We compute numerically a version of the model parameterized to match several features of the US economy. We show that, relative to self-insurance, Markov-perfect contracts provide substantial additional insurance, particularly for low-wealth

[9] More generally, if we solve the bargaining problem for an exogenously given agent's outside option $B(a)$ satisfying Assumption 2, then we would also obtain (12) as $\theta \to 0$.

[10] We thank an anonymous referee for pointing out this connection.

agents. We argue that a significant fraction of observed wealth inequality could be explained by introducing such contracts in the basic self-insurance framework. We further find that the welfare gains from resolving the commitment friction are larger than those from resolving the moral hazard problem at low asset levels, while the opposite holds at high asset levels. Finally, we show that the welfare gains associated with asset contractibility can be sizable when insurers have high market power.

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Appendix A. Proofs

Appendix A.1. Proposition 1

(i) Let $y_t \in \{y^1, ..., y^n\}$ be the realization of output in period $t \geq 0$ and let $s^t \equiv \{y_0, y_1, \ldots, y_t\}$ denote the history of output states up to time t. Let $\eta(s^t)$ denote the probability of output history s^t defined recursively as $\eta(\{s^{t-1}, y^i\}) = \pi_i \eta(s^{t-1})$, with $\eta(s^{-1}) = 1$ for $i = 1, ..., n$.

The first-best insurance contract $\{\tau^*(s^t), a^*(s^t)\}_{t=0}^{\infty}$ solves the problem,

$$\max_{\{\tau(s^t), a(s^t) \geq 0\}} \sum_{t \geq 0, s^t} R^{-t} \eta(s^t)[y_t - \tau(s^t)] \tag{A.1}$$

subject to the agent's ex-ante participation constraint

$$\sum_{t \geq 0, s^t} \beta^t \eta(s^t) u(c(s^t)) - B(a_0) \geq 0, \tag{A.2}$$

where $c(s^t) \equiv ra(s^{t-1}) + \tau(s^t) - a(s^t)$ is agent's consumption and $a(s^{-1}) = a_0 \in \mathbb{A}$ are the agent's initial assets. We can rewrite the above problem in a mathematically equivalent way in terms of $\{c(s^t), a(s^t)\}_{t=0}^{\infty}$ as:

$$\max_{\{c(s^t), a(s^t) \geq 0\}} \sum_{t \geq 0, s^t} R^{-t} \eta(s^t)[y_t - c(s^t)] - \sum_{t \geq 0, s^t} R^{-t-1} \eta(s^t) a(s^t)[R - r] + ra_0 \tag{A.3}$$

subject to (A.2). Since $r \leq R$ and given $a(s^t) \geq 0$, it is clearly optimal to set $a^*(s^t) = 0$ for any $s^t, t \geq 0$. Intuitively, 'extracting' agent's assets at time zero is optimal (strictly if $r < R$) since allowing the agent to carry assets over time at the inferior return r destroys surplus.

(ii)-(iii) To simplify notation, let $a^i_{t+1} \equiv a(s^t)$ and $c^i_t \equiv c(s^t)$ when $y_t = y^i$, for $i = 1, \ldots, n$. The first-order condition of problem (A.3) with respect to c^i_t is

$$-R^{-t} + \beta^t \lambda u_c(c^i_t) = 0, \tag{A.4}$$

where λ is the multiplier on the participation constraint. Equation (A.4) implies full insurance: $c^*(s^{t-1}, y^i) = c^*_t$ for all t, s^{t-1}, $i = 1, \ldots, n$—part *(ii)*. Given this, we obtain,

$$u_c(c_t) = \beta R u_c(c_{t+1}), \tag{A.5}$$

which yields part *(iii)*.

(iv) Suppose, at the first-best contract, the agent wanted to save an extra unit of resources privately. His first-order individual optimality condition in savings $a_{t+1} \geq 0$ is $u_c(c_t) \geq \beta r u_c(c_{t+1})$ with equality when $a_{t+1} > 0$. The agent's cost of saving an extra unit in today's utility is thus $u_c(c^*_t)$ which equals $\beta R u_c(c^*_{t+1})$ by (A.5), while his gain tomorrow is $\beta r u_c(c^*_{t+1})$. Since $R \geq r$, the cost is weakly larger than the gain. Thus, the agent would never wish to save privately if offered the first-best allocation, $\{c^*_t\}_{t=0}^{\infty}$. ■

Appendix A.2. Lemma 1

Since transfers enter linearly in the insurer's objective (4) and decrease his profits, the agent's participation constraint (5) always binds. If $v(a) > B(a)$ then the principal is necessarily, at some point of the contractual relationship, leaving the agent with some of the surplus. Thus, the participation constraint would not bind at some point, which contradicts the above statement. Hence, $v(a) = B(a)$ for all $a \in \mathbb{A}$. ■

Appendix A.3. Proposition 2

(i) Let $a = 0$ and guess $a' = 0$, that is, $c = c' = \mathcal{C}(0)$. Since $r - R + \frac{\beta B_a(0)(\beta R - 1)}{u_c(c)} < 0$ by Assumption 1, (11) implies $\zeta > 0$, i.e, $a' = \mathcal{A}(0) = 0$ indeed. Consumption at $a = 0$ is solved from (8). Our assumptions on B imply $\mathcal{C}(0) > 0$. Now consider any $a \in \mathbb{A}$ for which the optimal a' is $a' = 0$. From (8) we have $u(c) + \beta B(0) = B(a)$. The right-hand side is strictly increasing in a; thus, it must be that $u(c)$ increasing in a and so, by the continuity of u and B, consumption, $\mathcal{C}(a)$, is continuous and increasing in assets for all a with optimal $a' = 0$. We now show that the set of such a is an interval of the form $[0, \hat{a}]$.

Holding $a' = 0$ fixed, have from (11):

$$r - R + B_a(0)\left[\frac{\beta R}{u_c(\mathcal{C}(a))} - \frac{1}{u_c(\mathcal{C}(0))}\right] \leq 0. \tag{A.6}$$

From the discussion above, (A.6) holds with strict inequality at $a = 0$. Increase a away from zero infinitesimally and guess $a' = 0$ again. By the continuity of u, B and $\mathcal{C}$, since (A.6) is strict inequality at $a = 0$, the inequality must remain strict for $a = \varepsilon$ with $\varepsilon > 0$ and small enough, so we have a corner solution $a' = 0$ indeed. Next, since $c = \mathcal{C}(a)$ was shown to be continuous and increasing for any a with $a' = 0$ and since u_c is strictly decreasing, the left-hand side of (A.6) is increasing in a. Thus, by continuity there exists a unique positive value for a, which we call, $\hat{a}$ and at which (A.6) is satisfied with equality, while for any $a \in [0, \hat{a})$, (A.6) is satisfied with strict inequality, $a' = \mathcal{A}(a) = 0$. The Inada conditions on u guarantee $\mathcal{C}(\hat{a}) < \infty$ and thus, $\hat{a} < \infty$. For any $a > \hat{a}$ (A.6) is not satisfied (the left hand side becomes positive), which implies that $a' = 0$ is not its solution; thus, $\mathcal{A}(a) > 0$ for all $a > \hat{a}$. We prove that $\mathcal{A}(a) < a$ for $a > \hat{a}$ in part (iii) below.

(ii) In part *(i)* we showed that c is increasing in a for the case of a corner solution ($a' = 0$); thus $u_c(c) < u_c(c') = u_c(\mathcal{C}(0))$ for any $a \in (0, \hat{a}]$. In the case of interior solution, $a > \hat{a}$, (11) can be rearranged as

$$\frac{1}{u_c(c)} = \frac{1}{\beta R u_c(c')} + \frac{R - r}{\beta R B_a(a')}.$$

Given $B_a > 0$ and Assumption 1 ($R \geq r$ and $\beta R \leq 1$ but both do not hold at equality simultaneously), the above equation implies $u_c(c) < u_c(c')$. Thus, starting at any $a > 0$ the agent's consumption decreases over time.

(iii) The line of proof is as follows. We first use a duality argument to show that statically, higher assets give the agent incentives to smooth by picking both higher current consumption and future savings. The strict concavity of u and B is key for this result. We then show that the same intuition survives when the agent's future value from savings is included.

We will use the following auxiliary lemma.

Lemma A1: *Suppose we have the static "utility maximization" problem:* $\max_{c_1, c_2} \tilde{u}(c_1, c_2) = \phi^1(c_1) + \phi^2(c_2)$ *subject to* $p_1 c_1 + p_2 c_2 = m$, *with* ϕ^1, ϕ^2 *differentiable, strictly increasing and strictly concave. Then,* c_1 *and* c_2 *are normal goods for a consumer with these preferences.*

Proof of Lemma A1. The fact that additively separable utility with strictly concave sub-components implies normality is well-known (e.g., see Liebhafsky, 1969) ■

In part *(i)* we already showed that, at a corner solution for a', consumption is increasing in assets a. Consider now the case of an interior a'. First, we show that $\mathcal{C}(a)$ is increasing in a. Define $\tilde{\Pi}(a) \equiv \Pi(a) - ra$ and rewrite the insurer's problem (7) as:

$$\tilde{\Pi}(a) = \min_{c, a'} \; c + a'\left[1 - \frac{r}{R}\right] - \frac{\tilde{\Pi}(a')}{R} - \bar{y} \tag{A.7}$$

subject to

$$u(c) + \beta B(a') = B(a).$$

By (10) and the fact that B and u are increasing, $\tilde{\Pi}_a(a) < 0$, i.e., $\tilde{\Pi}(a)$ is a decreasing function.

Now look at the modified auxiliary problem $\min_{c,a'} \; c + a'(1 - \frac{r}{R})$ subject to $u(c) + \beta B(a') = z$. This can be viewed as a standard static "expenditure minimization" problem for a consumer with utility function $\tilde{u}(c, a') = u(c) + \beta B(a')$, which is the dual to the utility maximization problem in Lemma A1. Given the concavity of u and B, Lemma A1 implies that the optimal c and a' must go up with z ($= B(a)$), that is, when a increases. The same property must be true for c if we add the term $-\tilde{\Pi}(a')/R$ to the objective—since $\tilde{\Pi}$ is decreasing in a' and B is strictly concave, it is optimal to lower a' relative to what one would choose statically to trade present vs. future gains (alternatively, compare the first-order conditions of these two problems with respect to a'). But then c must increase relative to before to satisfy the constraint.

We next show that $a' = \mathcal{A}(a)$ is increasing in a. To see this, look at (11) at an interior solution:

$$r - R + B_a(a')\left[-\frac{1}{u_c(c')} + \frac{\beta R}{u_c(c)}\right] = 0. \tag{A.8}$$

Suppose $a' = \mathcal{A}(a)$ is non-increasing in a. Then, given that $\mathcal{C}(a)$ is increasing, the concavity of B and u, and our results from part *(i)*, we obtain a contradiction since the terms $B_a(a')$, $\frac{\beta R}{u_c(c)}$ and $-\frac{1}{u_c(c')}$ on the left-hand side of the above equation must be all increasing in a while the right-hand side is constant. Thus, $\mathcal{A}(a)$ is strictly increasing at an interior solution.

The result $a' = \mathcal{A}(a) < a$ (assets decrease over time) from part *(i)* follows immediately from our result that $\mathcal{C}$ is increasing in assets and part *(ii)*, which showed that consumption is decreasing over time, $\mathcal{C}(\mathcal{A}(a)) < \mathcal{C}(a)$.

(iv) We will show that for any $a_0 \in \mathbb{A}$ the sequence $\{a_k\}$ with elements $a_k \equiv \mathcal{A}(a_{k-1})$ for all $k \geq 1$ defining the time profile of agent's assets converges to zero in finite time. Suppose first $a_0 \in [0, \hat{a}]$. Then, by part (i) we have $a_1 = a_2 = ... = 0$ and hence agent's assets converge to zero in one step.

Let now $a_0 > \hat{a}$. The sequence $\{a_k\}_{k=1}^{\infty}$ defined above is monotonically decreasing (since $a_k = \mathcal{A}(a_{k-1}) \leq a_{k-1}$ by part (i)) and bounded (all its elements are non-negative). Thus, it is convergent. Call its limit $\tilde{a}$, i.e., $\mathcal{A}(\tilde{a}) = \tilde{a}$. If $\tilde{a} > \hat{a}$ we obtain a contradiction since it must be $\mathcal{A}(\tilde{a}) < \tilde{a}$ by part (i). So it must be that $\tilde{a} \in [0, \hat{a}]$. But then $\mathcal{A}(\tilde{a}) = 0$ by part (i), so it must be $\tilde{a} = 0$. Thus far, we have shown that agent's assets converge to zero.

We next show that the convergence must be in finite time. The definitions of convergent sequence and its limit applied to $\{a_k\}$ and $\tilde{a}$ imply that, for any $\varepsilon > 0$, there exists a natural number N such that for $n \geq N$ we have $|a_n - \tilde{a}| < \varepsilon$. Take some $\varepsilon > 0$ such that $\varepsilon < \hat{a}$. Then, there exists an N, possibly dependent on ε, such that $a_n < \varepsilon < \hat{a}$ for any $n \geq N$ (here we used that $\tilde{a} = 0$ and $a_n \geq 0$). Take $n = N$, for example. This implies assets converge to zero in at most $N + 1$ steps since $a_{N+1} = \mathcal{A}(a_N) = 0$ given that $a_N < \hat{a}$.

Finally, we show positive consumption in the long-run. Using (8), long-run consumption, $\mathcal{C}(0)$, solves $u(\mathcal{C}(0)) = (1 - \beta)B(0)$ and thus, by our assumptions on B, $\mathcal{C}(0) > 0$. ■

Appendix A.4. Proposition 3

(i) Let $r = R$. Define $\hat{\Pi}(a) \equiv \Pi(a) - Ra$ and rewrite the Markov-perfect contracting problem (7)—(8) as:

$$\hat{\Pi}(a) = \max_{c,a'} \ \bar{y} - c + \frac{\hat{\Pi}(a')}{R} \tag{A.9}$$

subject to

$$\begin{aligned} u(c) + \beta B(a') - B(a) = 0 \\ a' \geq 0. \end{aligned}$$

Take the second-stage problem under one-sided commitment, (14) and do a change of variables from w to a by calling $a = B^{-1}(w)$ for any w in the range of B (by our assumptions B is strictly monotonically increasing and so invertible) and calling $\bar{\Pi}(a) \equiv \tilde{\Pi}^{OS}(B(a)) = \tilde{\Pi}^{OS}(w)$. With this change of variables, problem (14) for state $w = B(a)$ is exactly the same mathematical problem as the MPE problem for state a, (A.9).[11] That is, any solution to the former problem is solution to the latter and vice versa. Consequently, $\bar{\Pi}(a) = \hat{\Pi}(a)$.

To finish the argument, look now at the first-stage of the one-sided commitment problem, (13) starting from initial assets a_0. Changing variables from w to a', by calling $a' = B^{-1}(w)$, and plugging in for $\tilde{\Pi}^{OS}(w) = \bar{\Pi}(a') = \hat{\Pi}(a')$, the first-stage problem (13) is equivalent to:

$$\max_{c_0,a'} \ \bar{y} - c_0 + \frac{\hat{\Pi}(a')}{R} + ra_0 \tag{A.10}$$

subject to

$$\begin{aligned} u(c_0) + \beta B(a') - B(a_0) = 0 \\ B(a') - B(0) \geq 0. \end{aligned}$$

[11] Note that the non-negativity constraint on assets, $a' \geq 0$, can be written as $B(a') - B(0) \geq 0$ and so, after changing variables, as $w - B(0) \geq 0$.

Since B is strictly increasing, the last constraint is equivalent to $a' \geq 0$, and so problem (A.10) has the same solution c_0, a' as the Markov-perfect problem (A.9) starting from the same a_0. In sum, the one-sided commitment and Markov-perfect problems with contractible assets initialized at the same asset level a_0 are equivalent to each other and hence, the consumption paths and time-zero profits (adding back the Ra_0 term to $\hat{\Pi}(a_0)$) generated by them coincide. The equivalence implies a one-to-one mapping between promised utility in one-sided commitment and asset levels in a Markov-perfect contract. If $R \neq r$ the equivalence obtains only if $a_0 \in [0, \hat{a}]$, i.e., if when no assets are carried by the agent in a MPE (see below for details).

(ii) Suppose now $r < R$. Define $\Pi^*(a) \equiv \Pi(a) - Ra$ and re-write the dynamic insurance problem with lack of commitment, (7)—(8) as:

$$\Pi^*(a) = \max_{c, a' \geq 0} \bar{y} - (R - r)a - c + \frac{\Pi^*(a')}{R} \tag{A.11}$$

subject to

$$u(c) + \beta B(a') - B(a) = 0.$$

Using the same arguments as in part (i), the second-stage one-sided commitment problem (14) does not depend on r and so remains the same as in part *(i)* i.e., equivalent to problem (A.9). Hence, when $r < R$ and $a > 0$, the problem with no commitment, (A.11) is no longer equivalent to problem (A.9) (which, remember, is equivalent to the second-stage one-sided commitment problem) because of the extra term $(R - r)a$ in the objective. In the special case $a_0 \in [0, \hat{a}]$ we know from Proposition 2 that $a_1 = a' = 0$ and hence agent's assets stay at zero forever after the initial period. Thus, going to the second-stage problem with assets $a = 0$, the extra term $(R-r)a$ drops out restoring the equivalence between the one-sided commitment problem and the Markov-perfect insurance problem.

More specifically, let $\{c_t^M, a_{t+1}^M\}_{t=0}^\infty$ solve problem (A.11) starting from assets a_0 and following the MPE policies $\mathcal{C}$ and $\mathcal{A}$. By repeatedly plugging in, we can write its $t = 0$ present discounted value as:

$$\Pi^*(a_0) = \sum_{t=0}^{\infty} R^{-t} \left[\bar{y} - c_t^M - (R - r)a_t^M\right]$$

where $a_0^M = a_0$.

Using the equivalence result from part (i), in particular problems (A.10)–(A.9), the time-zero value of the one-sided commitment problem's objective function evaluated at our MPE solution from a_0 is $ra_0 + \sum_{t=0}^\infty R^{-t}[\bar{y} - c_t^M]$. Since the Markov-perfect and one-sided commitment problems share the same set of constraints, the sequence $\{c_t^M, a_{t+1}^M\}_{t=0}^\infty$ is feasible for the one-sided commitment problem from a_0 but not necessarily optimal. Thus, letting $\bar{\Pi}(a_0)$ be the maximized value of the one-sided commitment contract we have,

$$\bar{\Pi}(a_0) \geq ra_0 + \sum_{t=0}^{\infty} R^{-t} \left[\bar{y} - c_t^M\right] \geq Ra_0 + \sum_{t=0}^{\infty} R^{-t} \left[\bar{y} - c_t^M - (R - r)a_t^M\right] = \Pi(a_0),$$

Since $R > r$ the second inequality would be strict if at least one a_t^M is strictly positive for some $t \geq 1$. Using Proposition 2(i), this implies that for any $a_0 > \hat{a}$ MPE profits, $\Pi(a_0)$ are strictly lower than profits with one-sided commitment, $\bar{\Pi}(a_0)$ since for $a_0^M = a_0 > \hat{a}$ at least $a_1^M > 0$ (and possibly other a_t^M with $t > 1$ are also positive).

For part (b) use the first-order conditions of the one-sided commitment problem, (13) and (14) to obtain $u_c(c) = \beta R u_c(c')$ at an interior solution. This coincides with the Euler equation in the first-best, (A.5). Thus, our discussion in Section 3.1 below equation (12) implies that consumption is reduced faster in a MPE than under one-sided commitment. When $\beta R < 1$ both respective Euler equations, (A.5) and $u_c(c) = \beta R u_c(c')$ in the Markov-perfect and one-sided commitment settings imply that consumption is decreasing over time. From the agent's participation constraint, long-run consumption must be the same in both settings—the value $c^* > 0$ solving $u(c^*) = (1 - \beta)B(0)$. ■

Appendix A.5. Lemma 2

For any a such that $a'^N > 0$ (an interior solution), (19) and the strict concavity of u and B imply that c^N and a'^N must move in the same direction as a varies. In addition, since u and B are increasing, (18) implies that it is not possible that c^N and a'^N both decrease in a. For any remaining a for which we do not have interior solution (if such exist), constraint (19) does not bind and the previous arguments from Proposition 2 apply. ■

Appendix A.6. Proposition 4

Suppose Markov-perfect contracts with contractible and non-contractible assets coincide for all a. Then, the contractible-assets case contracts $\{c = \mathcal{C}(a), a' = \mathcal{A}(a)\}$ solving problem (7)–(8) must satisfy constraint (15) in the non-contractible assets problem for any a. Recall the first-order condition with respect to a' in the contractible-assets case, $\Pi_a(a')/R + \lambda[-u_c(c) + \beta B_a(a')] + \zeta = 0$, and take some $a > \hat{a}$ (i.e., where $\zeta = 0$ and $a' > 0$). If $\Pi_a(a') < 0$ for that a, then (since $\lambda = 1/u_c > 0$) we have $-u_c(c) + \beta B_a(a') > 0$, which contradicts (15). Conversely, if $\Pi_a(a') > 0$, then $-u_c(c) + \beta B_a(a') < 0$ and so (15) implies $a'^N = 0$, which differs from $a' > 0$. Thus, in both these cases MPE contracts with and without asset contractibility differ. The only remaining possibility is $\Pi_a(a') = 0$ for all $a > \hat{a}$ which, by Proposition 2(i), is equivalent to $\Pi_a(a) = 0$ for all $a \in (0, \mathcal{A}(\bar{a})]$. Assume $\bar{a}$ is chosen sufficiently large so that $\mathcal{A}(\bar{a}) > \hat{a}$ (this is always possible by Lemma 2) and suppose $\Pi(a)$ is constant for all such a. Take some $a_1 > \hat{a}$ and call its corresponding optimal contract $\{c_1, a_1'\}$. Let $a_2 \equiv a_1 + \varepsilon$ for some small $\varepsilon > 0$ and let our Markov-perfect contract for a_2 be $\{c_2, a_2'\}$. From (10) $\Pi_a(a) = 0$ implies $ru_c(c) = B_a(a)$, and so the contract $\{c_1 + r\varepsilon, a_1'\}$ satisfies the participation constraint at a_2. That is, this contract is feasible at a_2 but not profit-maximizing (e.g., since we know $a_1' < a_2'$ by Proposition 2(iii)). This implies that $\Pi(a_2)$, the maximized profit at a_2, is larger than the profit which is achieved by the sub-optimal contract $\{c_1 + r\varepsilon, a_1'\}$ at a_2, i.e.,

$$\begin{aligned}\Pi(a_2) &= \bar{y} + ra_2 - c_2 - a_2' + \frac{\Pi(a_2')}{R} \\ &> \bar{y} + ra_2 - c_1 - r\varepsilon - a_1' + \frac{\Pi(a_1')}{R} \\ &= \bar{y} + ra_1 - c_1 - a_1' + \frac{\Pi(a_1')}{R} = \Pi(a_1).\end{aligned}$$

In sum, $\Pi(a_2) > \Pi(a_1)$—a contradiction with the assumption that $\Pi(a)$ is constant. Thus, there exists an $a > \hat{a}$ for which $\Pi_a(\mathcal{A}(a)) \neq 0$. For any such a Markov-perfect contracts with and without assets contractibility differ.

To show the remainder of part *(ii)*, note that the first-order condition of problem (7) for any a for which $a' > 0$ implies

$$\Pi_a' R^{-1} + \lambda\left[-u_c(c) + \beta B_a(a')\right] = 0 \tag{A.12}$$

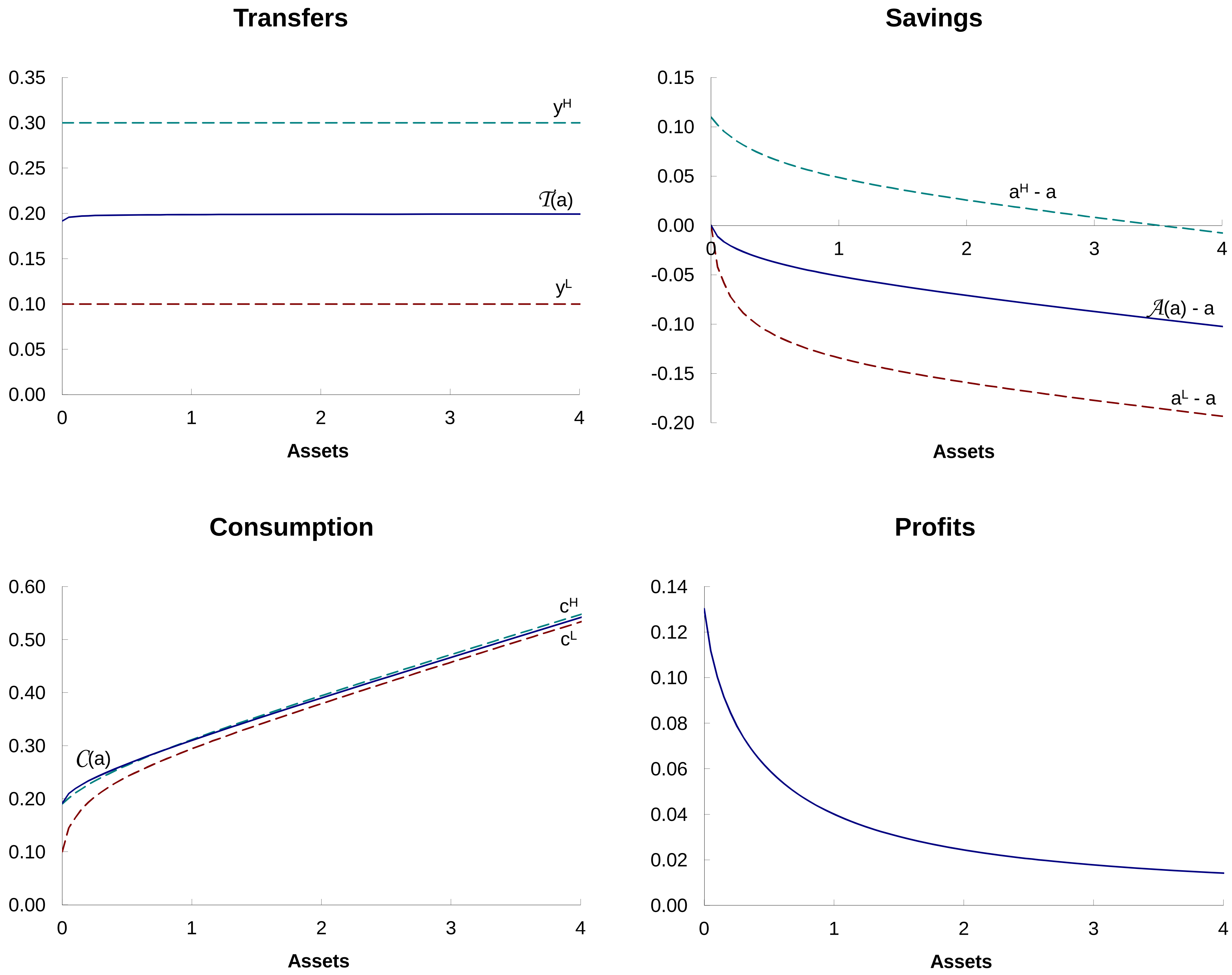
Suppose $\Pi_a(a') < 0$. Since $\lambda > 0$, (A.12) implies $u_c(c) < \beta B_a(a')$. In contrast, at an interior solution in problem (17) we have $u_c(c^N) = \beta B_a(a'^N)$. Hence, for any such a, when assets are contractible the agent saves strictly less (and consumes more) compared to in the non-contractible assets case, that is, $c^N < c$ and $a'^N > a'$. The opposite is true if $\Pi_a(a') > 0$. ■

Appendix A.7. Proposition 5

(i) The first-order condition of the insurer's problem with respect to assets is $-u_c(c) + \beta v_a(a') + \xi = 0$. The envelope condition implies $v_a(a) = ru_c(c)$. At an interior solution, $\xi = 0$, and thus, $u_c(c) = \beta ru_c(c')$.

(ii) For property (i) of Proposition 2, follow the same approach as in the proof of that proposition. Let $a = 0$ and guess $a' = 0$. By Assumption 1, $-u_c(\bar{y}) + \beta ru_c(\bar{y}) < 0$, which, using the first-order and envelope conditions implies $\xi > 0$. Continue as in Proposition 2. Naturally, the threshold $\hat{a}$ may be different here. Property (ii) follows directly from the Euler equation, (23). The proof of property (iii) follows the same approach as in the proof of Proposition 2, using Lemma A1 for the auxiliary problem $\max_{c,a'} u(c) + \beta v(a')$ s.t. $c + a' = ra + \bar{y}$. Property (iv) follows directly from properties (i) and (ii). The positive long-run consumption level is $c = \bar{y}$.

(iii) Suppose an agent is offered $\mathcal{T}(a) = \bar{y}$ corresponding to MPE with contractible assets, but we allow him to choose his own savings, a'. The first-order condition of the agent's problem implies (at an interior solution) $-u_c(\bar{y}+ra-a')+\beta v_a(a') = 0$. Since transfers $\mathcal{T}(a)$ do not depend on assets, the envelope condition is $v_a(a) = ru_c(\bar{y} + ra - a')$, which implies $u_c(c) = \beta ru_c(c')$—the same condition as in *(i)*. ■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yL
Savings
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0.10
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-0.20
0 1 2 3 4
Assets
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𝒜(a) - a
aL - a
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cL
𝒞(a)
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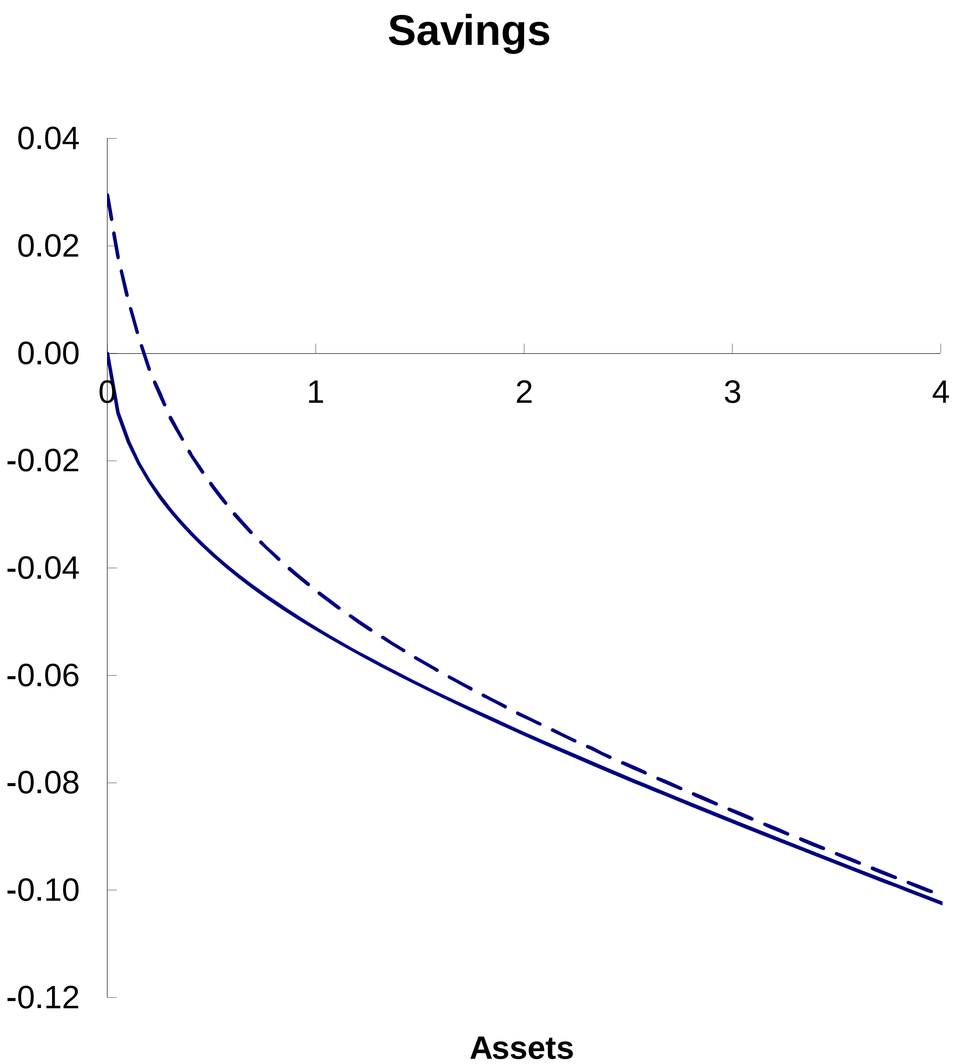
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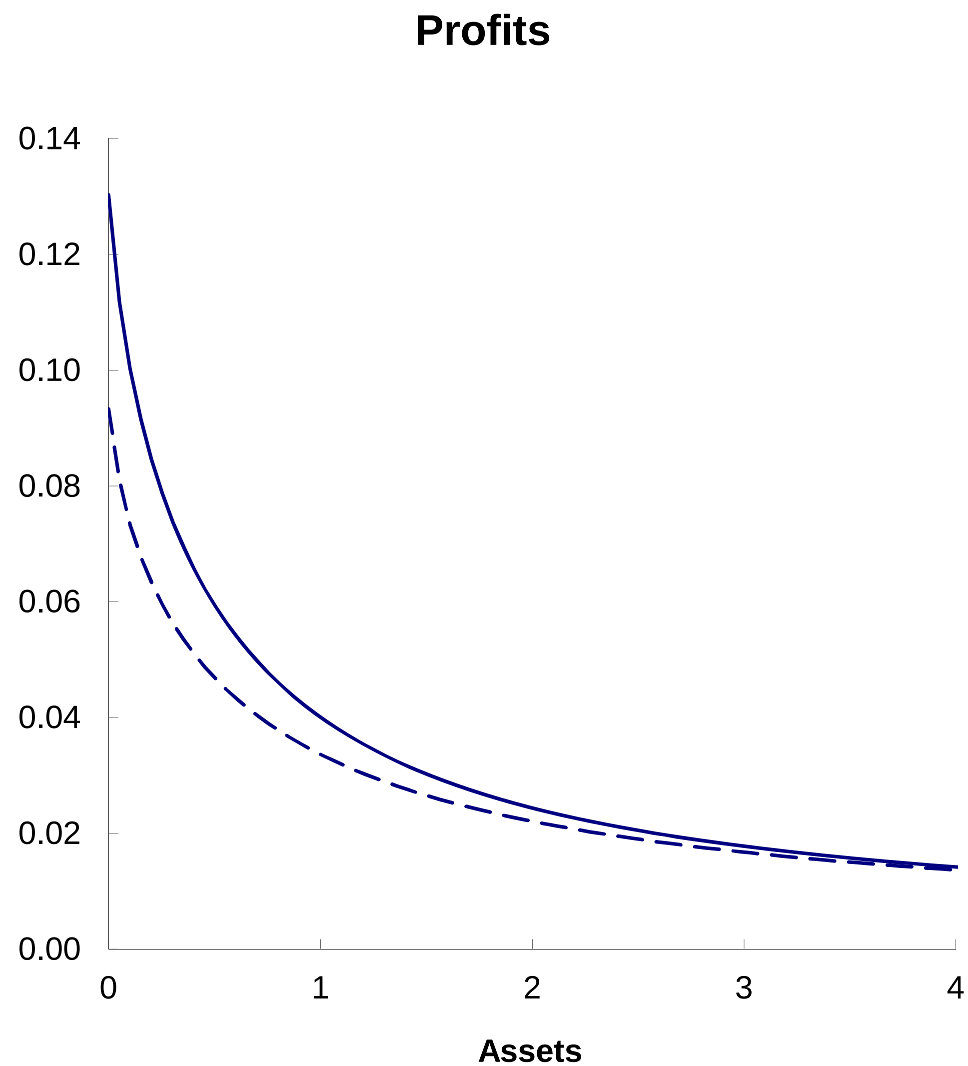
Assets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