The odds-ratio: what it is and why it should be used with caution

Martin A. Andresen
andresen@sfu.ca

School of Criminology, Simon Fraser University
8888 University Drive, Burnaby, BC V5A 1S6 Canada

and

Institute for Canadian Urban Research Studies
Simon Fraser University

November 2009

Abstract

The commonly used odds-ratio in the public health and medical literatures is too often misinterpreted or represented in incomplete form. Deriving the mathematics behind the odds-ratio, it is shown that if the odds-ratio is only partially represented, in its form of pure risk, it has almost zero utility in health policy and research. The odds-ratio must be used cautiously and interpreted in a manner that properly represents research findings in order to avoid potentially false inference.

Keywords: odds-ratio; statistical inference; relative risk

Introduction

The odds-ratio is one of the most common forms of representing risk, but is too often not represented in a complete manner in both the top public health[1,2] and medical[3,4,5] literatures. This misrepresentation potentially leads to misinformed inference and public health policy. Despite research showing how the odds-ratio must be transformed to properly assess risk,[6,7,8] which is independent of the point below, little evidence of this transformation has emerged. A recent paper[9] has been instructive in showing the discrepancies between relative risk and the odds-ratio, but greater detail in the calculation of the odds-ratio aids in understanding the misrepresentations of risk using this measure.

The misrepresentation of risk is, admittedly, partially a misnomer. The difficulty arises in the interpretation of relativity. In order to understand the interpretation of relativity on the context of the odds-ratio, the mathematics behind the ratio need to be explored.

The Mathematics of the Odds-Ratio

The odds-ratio is calculated using the logistic function:

$$\Pr(Y=1) = {\exp(X\beta)}\big/{(1+\exp(X\beta))} \qquad \Pr(Y=0) = {1}\big/{(1+\exp(X\beta))}.$$

To simplify notation let $\Pr(Y = 1) = p$ and $\Pr(Y = 0) = 1 - p$, $p \in (0,1)$, which allows the odds-ratio to be calculated:

$$\text{Odds-Ratio} = \frac{p}{1-p} = \frac{\exp(X\beta)/(1+\exp(X\beta))}{1/(1+\exp(X\beta))} = \exp(X\beta);\ \frac{p}{1-p} \in (0,\infty)\,.$$

Taking the natural logarithm of this equation:

$$\ln\left(\frac{p}{1-p}\right) = X\beta;\ \ where\ X\beta = X_0\beta_0 + X_1\beta_1 + X_2\beta_2 + \cdots + X_n\beta_n;\ \ and\ X_0 = 1,$$

and the derivative to get the relationship between the estimated parameters and the odds-ratio:

$$\frac{\partial \ln\left(\frac{p}{1-p}\right)}{\partial x_i} = \beta_i \quad or\ given\ \Delta x_i,\ \Delta \ln\left(\frac{p}{1-p}\right) = \beta_i$$

it is shown that the parameter, β_i, represents the percentage change in the log-odds ratios from a unit change in one of the independent variables, x_i. The question, then, is whether a percentage change in the log-odds ratio translates into an equivalent, or equal, change in the probability.

The first step is to establish the relationship between the odds-ratio and the probability, p.

Log-Odds = $\ln(p/(1 - p)) = \ln(p) - \ln(1/(1 - p))$

$$\frac{\partial(\log - \text{odds})}{\partial p} = \frac{1}{p} - \frac{1}{(1-p)^2} \frac{<}{>} 0$$

Therefore, the relationship between the odds-ratio and the probability, *p*, cannot be known *a priori*. But, does a 10 percent change in the log-odds ratio tell us anything about the change in probability? The short answer is yes, but in a very particular way.

Expanding on the term representing percentage change in the log odds-ratio:

$$\Delta \ln\left(\frac{p}{1-p}\right) = \ln\left(\frac{p_2}{1-p_2}\right) - \ln\left(\frac{p_1}{1-p_1}\right) = \beta_i ,$$

where p_1 and p_2 are the two probabilities defining the change in the log-odds ratio. Recalling that:

$$\ln\left(\frac{p_2}{1-p_2}\right) - \ln\left(\frac{p_1}{1-p_1}\right) = \ln\left(\frac{p_2 / (1-p_2)}{p_1 / (1-p_1)}\right) \text{ and } \exp(\ln x) = x,$$

it is shown that $\frac{p_2 / (1-p_2)}{p_1 / (1-p_1)} = \exp(\beta_i)$ and the usually reported odds-ratio in the medical literature is actually a ratio of two odds-ratios.

To simplify notation, $P_i = {p_i}/{(1-p_i)}$, recalling from above $P_i \in (0,\infty)$, $i = 1, 2$. Also note that $\exp(\beta_i) \in (0,\infty)$, $\forall\ \beta_i$. This simplifies the above equation to the following:

$$\frac{P_2}{P_1} = \exp(\beta_i)\,.$$

Given a $\exp(\beta_i)$, is there a unique probability? Solving for P_2: $P_2 = P_1 \exp(\beta_i)$. Because $P_1, P_2, \exp(\beta_i) \in (0,\infty)$ are all positive real numbers, for every $\exp(\beta_i)$ and for any initial odds-ratio, P_1, there exists another odds-ratio, P_2. And, because there are an infinite number of rational numbers between any two distinct real numbers, there are an infinite number of combinations of (P_1, P_2) that will satisfy $P_2 = P_1 \exp(\beta_i)$.

Example:

Suppose a) if $\exp(\beta_i) = 10$ and $P_1 = 10$, then $P_2 = 100$.

Suppose b) if $\exp(\beta_i) = 10$ and $P_1 = 0.0001$, then $P_2 = 0.001$.

Because the odds-ratio is increasing in p, the direction of change is correct, but there is no indication of magnitude, only relative change.

In a) $p_1 = 0.000099$ and $p_2 = 0.00099$, therefore $\Delta p = 0.089$ percent.

In b) $p_1 = 0.\ 909$ and $p_2 = 0.\ 99$, therefore $\Delta p = 8.1$ percent.

Therefore, for a particular $\exp(\beta_i)$ value, we can not only have two different probability changes, but two different probability changes that are different by two decimal places: one that has little change in probability and the other has (arguably) large change. Moreover, this is true for every possible value of $\exp(\beta_i)$. The end result is that the relative changes in probabilities are exactly the same, as stated in statistics textbooks that cover the odds-ratio,[10,11] but those *relative* changes say nothing for absolute change, which is very important in evaluating the effects of medical treatment. In other words, though used in a technically correct manner, sometimes, the odds-ratio leaves much to be desired.

Conclusion: Measuring Changes in Probability

This short paper has shown that the odds-ratio, as commonly employed in medical and public health contexts, has the potential for giving misleading results. As with any quantitative analysis, both the relative and absolute effects need to be accounted for to provide a complete interpretation. The way to avoid this difficulty with the odds-ratio is to directly calculate the change in probabilities from a change in an independent variable.[12] By performing this type of analysis the research is not only presented in a manner that assesses the relative effects of treatments on the probabilities, which are important, but also in a manner that reveals the actual magnitudes of those relative effects. Only when both the relative and the absolute are taken into account is the assessment of risk properly represented.

References

[1]Alavanja MCR, Hoppin JA, Kamel F. Health effects of chronic pesticide exposureL cancer and neurotoxicity. *Annual Review of Public Health* 2004; 25: 155 – 197.

[2]Engström K, Diderichsen F, Laflamme L. Parental social determinants of risk for intentional injury: a cross-sectional study of Swedish adolescents. *American Journal of Public Health* 2004; 94: 640 – 645.

[3]Leitzmann MF, Platz EA, Stampfer MJ, Willett WC, Giovannucci E. Ejaculation frequency and subsequent risk of prostate cancer. *Journal of the American Medical Association* 2004; 291: 1578 – 1586.

[4]Renehan AG, Zwahlen M, Minder C, O'Dwyer ST, Shalet SM, Egger M. Insulin-like growth factor (IGF)-I, IGF binding protein-3, and cancer risk: systematic review and meta-regression analysis. *The Lancet* 2004; 363: 1346 – 1353.

[5]Yudkin P, Munatò M, Hey K, Roberts S, Welch S, Johnstone E, Murphy M, Griffiths S, Walton R. Effectiveness of nicotine patches in relation to genotype in women versus men: randomized controlled trial. *British Medical Journal* 2004; 328: 989 – 990.

[6]Zhang J, Yu KF. What's the relative risk? A method of correcting the odds ratio in cohort studies of common outcomes. *Journal of the American Medical Association* 1998; 280: 1690 – 1691.

[7]Davies HTO, Crombie IK, Tavakoli M. When can odds ratios mislead? *British Medical Journal* 1998; 316: 989 – 991.

[8]Deeks J, Bracken MB, Sinclair JC, Davies HTO, Tavakoli M, Crombie IK. When can odds ratios mislead? *British Medical Journal* 1998; 317: 1155.

[9]Schmidt CO, Kohlmann T. When to use the odds ratio or the relative risk? *International Journal of Public Health* 2008; 53: 165 – 167.

[10]Hamilton LC. *Regression with Graphics: A Second Course in Applied Statistics* 1992; Belmont, CA: Duxbury Press.

[11]Tabachnick BG, Fidell LS. *Using Multivariate Analysis, 4th ed* 2001; Boston: Allyn and Bacon.

[12]Greene WH. *Econometric Analysis, 4th ed* 2000; Upper Saddle River, NJ: Prentice Hall.