

## Overlapping generations economies with growth

- agents live for two periods
- economy lasts for a finite or infinite number of periods,  $T$
- at each time  $t$ ,  $N(t)$  of generation  $t$  are born, an agent  $h$  of generation  $t$  lives through periods  $t$  and  $t + 1$

$$N(t) = (1 + n)N(t - 1)$$

- $n$  - rate of population growth
- single consumption good; agent  $h$ 's preferences are defined over consumption of the good at time  $t$  (when young),  $c_t^h(t)$ , and time  $t + 1$  (when old)  $c_t^h(t + 1)$ :

$$u_t^h(c_t^h(t), c_t^h(t + 1))$$

- if a unit of the consumption good is stored between periods  $t$  and  $t + 1$ , it becomes a unit of capital at time  $t + 1$ .

- agents are endowed with units of labor when young and old
- labor endowment pattern,  $l^h(t)$  of agent  $h$  of generation  $t$  is given by:

$$l_t^h = [l_t^h(t), l_t^h(t + 1)]$$

where  $l_t^h(t)$  are units of labor when young, and  $l_t^h(t + 1)$  are units of labor when old

- labor inelastically supplied
- total labor supply in the economy at time  $t$ ,  $L^s(t)$ , is fixed and given by:

$$L^s(t) = \sum_{i=1}^{N(t)} l_t^i(t) + \sum_{j=1}^{Nt-1} l_{t-1}^j(t)$$

- agents earn wages  $w(t)$  at time  $t$  (given in terms of the consumption good)
- there is technology for producing the consumption good which takes labor and capital as inputs

$$F(L, K)$$

- it is a constant returns to scale technology, i.e. if we double both inputs, the output doubles

- if capital is held constant, and labor keeps increasing, marginal product of labor decreases with additional labor (i.e. the additional amount of output that is obtained goes down)

- likewise, if labor is kept constant, and capital keeps increasing, marginal product of capital decreases with additional units of capital

- Cobb-Douglas production function

$$Y(t) = A(t)K(t)^\alpha L(t)^{(1-\alpha)}$$

-  $Y(t)$  - amount of output produced at time  $t$

-  $A(t)$  - technology factor, can be changing over time; a way to represent changes in technology; it can be changing according to some exogenously specified process, or endogenously, where it is then variable that depends on what happens in the economy

- later on, we will spend more time talking about different descriptions of  $A(t)$ ;

- for the current analysis, let us assume that  $A(t)$  is constant over time, i.e.

$$A(t) = A = 1$$

for all  $t$

$K(t)$  is a total amount of capital that is used in the production at time  $t$

- we will assume that the depreciation rate  $\delta$  is equal to 1

-  $L(t)$  is total amount of labor used at time  $t$

-  $0 < \alpha < 1$

- in equilibrium,  $\alpha$  - share of capitals income;  $(1 - \alpha)$  - share of labors income

-  $w(t)$  - wage that clears the labor market at time  $t$

- labor income, wage that is paid at time  $t$ ,  $l_t^h(t)w(t)$

- total labor income when old, at  $t + 1$ ,  $l_t^h(t + 1)w(t + 1)$

- capital income rental on invested capital at time  $t$ ,  $R(t)$

- profit-maximizing, competitive firms, producing the same type of the consumption good

- in equilibrium, firms make zero profits, i.e. all output is paid out to the factors of production

- in equilibrium:

$$MPL = (1 - \alpha)K^\alpha(t)L^{-\alpha}(t) = w(t)$$

$$MPK = \alpha K^{\alpha-1}(t)L^{1-\alpha}(t) = R(t)$$

- in addition to capital market, there is a private bonds market

-  $b^{p,h}(t)$  - bond holdings between  $t$  and  $t + 1$

-  $b^{p,h}(t) > 0$  - a lender

-  $b^{p,h}(t) < 0$  - borrower

- agent  $h$ 's savings at time  $t$  (can be  $< 0$  or  $> 0$  depending on  $b^{p,h}(t)$ )

-  $k^h(t + 1)$  - the amount of the consumption good that is stored until  $t + 1$  and used in the production at  $t + 1$

- savings decision of young agents at the end of the first period of their lives -  $s_t^h(t)$

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$$s^h(t) = b^{p,h}(t) + k^h(t + 1)$$

- savings invested as capital in the production in the following period,  $t + 1$

$$S(t) = \sum_{h=1}^N s_t^h(t)$$

- old agents consume their labor income (wages received if working when old) + capital income (earned by saving and investing in capital)

- agents maximize their utilities subject to lifetime constraints

- agents maximization problem:

$$\max_u^h(c_t^h(t), c_t^h(t + 1))$$

$$s.t. c_t(t) \leq w(t)s(t)$$

$$c(t+1) \leq s(t)R(t)$$

- derive aggregate savings function:

- in equilibrium, no arbitrage condition holds, i.e.  $R(t) = 1 + r(t)$  - as there is no uncertainty in the economy regarding the technology or the preferences, the rates of return on two assets (private loans and capital) have to be equal

$$S(t) = \sum_{h=1}^{N(t)} s^h(t) = \sum_{h=1}^{N(t)} b^{p,h}(t) + \sum_{h=1}^k (t+1)$$

**Market clearing conditions in the assets' markets**

$$B^p(t) = \sum_{h=1}^{N(t)} b^{p,h}(t) = 0$$

$$S(t) = K(t+1)$$

total amount of capital employed in the production

$S(t)$  - supply of capital for production at time  $t+1$

$K(t+1)$  - demand for capital by firms at time  $t+1$

**Equilibrium conditions:**

At each  $t$ , there are prices  $w(t)$ ,  $R(t)$ , and  $(1 + r(t))$  at which:

1. Agents maximize their utilities subject to lifetime constraints

2. Firms maximize their profits

3. All markets clear:

a. Goods market total demand for the good by the old and by the young equals the total amount produced

b. total supply of labor equal total demand for labor

c. Net lending/borrowing equals 0

d. Aggregate supply of savings equals firms demand for capital