The Theory of Choice: Utility Theory

(Based on Copeland and Weston, “Financial Theory and Corporate Policy”)

Finance can be regarded as a sub discipline of economics that focuses on no-arbitrage (the law of one price) in financial markets (see chapter 3 of Berk and Demarzo).

In Economics, however, researchers study how individual people and societies choose to allocate scarce resources and distribute wealth among one another and over time. In order to quantify how people are made better or worse off from a particular bundle of goods, a common tool is to use a utility function.

Example: suppose that a person is twice as much satisfied with apples than with bananas. We can define the individual utility function as \( U(a, b) = 2 \times a + b \), where \( a \) stands for the number of apples and \( b \) the number of bananas. Using such an individual utility function allows us to rank different combinations of apples and bananas for that individual. It also allows us to draw indifference curves, which represent different combinations of apples and bananas that provide the same utility for that person.

In Finance, we are concerned with one type of good, money (or wealth). Further, our main concern is about making decision under uncertainty (when we do not really know what the exact outcome will be). To this end, one would need tools that allow analyzing how people make choices under uncertainty.

Expected Utility

To develop a theory of rational decision making in the face of uncertainty, it is necessary to make some very precise assumption about an individual’s behaviour. Theory states that based on a few axioms of rational behaviour (comparability, transitivity, strong independence, measurability, and ranking), one can always find a utility function that allows the individual to make choices when faced with uncertainty.

Just as an example of one axiom: Transitivity means that if “I like Chevrolets more than Fords and Fords more than Toyotas then I must also like Chevrolets more than Toyotas, otherwise my preferences are not rational”. Anyway, based on the existence of rational behaviour, it is possible to prove that there exists a utility function that will have two properties that allows making choice under uncertainty:

1. If we measure the utility of \( x \) and find that it is greater than the utility of \( y \), \( U(x) > U(y) \), it means that \( x \) is actually preferred to \( y \), \( x \succ y \).
Expected utility can be used to rank combinations of risky alternatives. Mathematically, this means that, $U[G(x, y; \alpha)] = \alpha U(x) + (1 - \alpha) U(y)$, where $G(.)$ is the individual’s utility function for the gamble, $x$ and $y$ the two possible outcomes, and $\alpha$ and $(1 - \alpha)$ the probability that $x$ and $y$ will occur, respectively (note the sum of probabilities for the different outcomes is 1).

When there are more than two states, we can write the expected utility of wealth as follows: $E[U(W)] = \sum p_i U(W_i)$, where $p_i$ is the probability of ending at state $i$ and $W_i$ is the terminal wealth in state $i$.

We can now use the concept of expected utility to establish definition of risk premium and also risk aversion. As a starting point, we begin by assuming that more wealth is always better than less wealth (“more money is always good”). Mathematically stated, we are only interested in utility functions where wealth has a positive marginal utility (first derivative of the utility function is positive, $U'(W) > 0$).

Let the probability of receiving prospect $x$ be $\alpha$ and the probability of $y$ be $(1 - \alpha)$. The gamble can be written as before: $G(x, y; \alpha)$. Now the question is this: Does the individual prefer the expected value of the gamble (i.e., expected average outcome) with certainty, or does he prefer the gamble itself? For example, would the individual like to receive $5 for sure, or would she prefer to “roll the dice” in a gamble that pays off $8 with a 50% probability and $2 with a 50% probability?

A person who prefers the gamble to the expected value, we call a risk lover and for her, $U[E(W)] < E[U(W)]$.

A person who is indifferent between the expected value and gamble, we call a risk neutral and for her, $U[E(W)] = E[U(W)]$.

A person who prefers the expected value to the gamble we call a risk averse and for her, $U[E(W)] > E[U(W)]$. 

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The above graphs provide examples that represent utility functions for a risk lover, risk neutral, and risk averse individual, respectively. Note that whether the expected utility (i.e., $E[U(W)]$) falls below (or above) the value of the utility function for the average payoff of $5 (i.e., U(5)) or not is crucial for determining the risk attitude of the individual.
Note that if our utility function is strictly concave, the individual is risk averse. Mathematically, a risk-averse individual has a utility function whose second derivative is negative $U''(W) < 0$. It is even possible to compute the maximum amount of wealth an individual would be willing to give up in order to avoid the gamble.

**Example:** Suppose Mr. Smith has a log utility and is faced with the above gamble. His utility from $5 is $0.5 \times \log(5) = 0.70$, while the expected utility from the gamble is $0.5 \times \log(8) + 0.5 \times \log(2) = 0.60$. This is not surprising as we know that Mr. Smith is a risk averse individual as he has a log utility function whose second derivative is negative note that $U''(\log(W)) = -\frac{1}{W^2} < 0$.

In fact we can figure out what is the certain sum of money that makes Mr. Smith have a utility of 0.60.

\[
\log(X) = 0.60
\]
\[
X = 10^{0.6} = \$3.98
\]

This would mean that compared to a certain amount of $5, Mr. Smith values the gamble to be only $3.98, or $1.02 less. In other words, Mr. Smith would be willing to pay up to $1.02 in order to avoid the gamble. If Mr. Smith was offered insurance against the gamble that costs less than $1.02, he will buy it.

**Quantifying Risk Premium**

The Risk premium is the difference between the individual’s expected wealth, given the gamble, and the level of wealth the individual would accept with certainty if the gamble were removed, i.e., his or her certainty equivalent wealth.

**Example:** A risk-averse individual has logarithmic utility and current wealth of $10, but the gamble is a 10% chance of winning $10 and a 90% chance of winning $100. What is the risk premium of the gamble?

Current wealth = $10

Expected wealth = $10 + 0.1 \times 10 + 0.9 \times 100 = 101

Certainty equivalent wealth: first we calculated expected utility of terminal wealth:

\[
U(E(W)) = 0.1 \times \log(20) + 0.9 \times \log(110) = 1.967
\]

Translate to certainty equivalent wealth -

\[
X = 10^{1.967} = \$92.76
\]
Risk premium = Expected wealth – certainty equivalent wealth = 101 - 92.76 = $8.24

Note that the gamble is still something that increases the wealth of the individual and he/she would be willing to pay money to enter such a gamble. The value of the gamble is the difference between the certainty equivalent wealth and the initial wealth (prior to the gamble). In our case, 92.67 - 10 = $82.76. This is the maximum amount that such an investor would be willing to pay to enter the gamble.

Quantifying Risk Aversion

In general, it is common to assume that all individuals are risk averse. Mathematically, that would mean that all utility functions are strictly concave and increasing with the following properties: $U'(W) > 0$, $U''(W) < 0$.

Now that we have learnt how to measure risk-premium, it would be interesting to provide a more specific definition of risk aversion. In other words, it would be nice if we could compare individuals in terms of their overall attitude towards risk. A nice way of interpretation of risk aversion is that a risk-averse individual does not like neutral gambles, where the expected payoff is zero. Thus, if we have a gamble whose payoff is on average zero, a risk-averse individual would not want to participate. Mathematically, a fair gamble can be expressed as a random variable $\tilde{Z}$ with $E(\tilde{Z}) = 0$. The tilda sign above the $Z$ means that the value of $Z$ is drawn out of a certain distribution.

Note that in the previous example, the risk premium of a gamble was a function of the level of wealth, $W$, and the gamble, $\tilde{Z}$. Mathematically, the risk premium, $\pi$, can be defined as the value that satisfies the following equality:

1. $E[U(W + \tilde{Z})] = U[W + E(\tilde{Z}) - \pi(W, \tilde{Z})]$

Working with the LHS, we can use some basic know-how and the Taylor’s series of expansion to get the following:

$LHS: \quad U[W + E(\tilde{Z}) - \pi(W, \tilde{Z})] = U[W - \pi(W, \tilde{Z})] = U(W) - \pi U'(W) + small \ terms$

Using Taylor’s series of expansion for the LHS, we get.

$RHS: \quad E[U(W + \tilde{Z})] = E \left[ U(W) + \tilde{Z} U'(W) + \frac{1}{2} \tilde{Z}^2 U''(W) + small \ terms \right]$

After taking expectation, we get, $U(W) + \frac{1}{2} \sigma_z^2 U''(W) + small \ terms$

Next, equating the RHS with the LHS, and solving for $\pi$, we get
\[ \pi = \frac{1}{2} \sigma_z^2 \left( -\frac{U''(W)}{U'(W)} \right) \]

This is a measure of a local risk premium. Since \( \frac{1}{2} \sigma_z^2 \) the variance is always positive, the sign of the risk premium is always determined by the sign of the term in parenthesis, and the risk premium will be positive when we have \( U'(W) > 0, U''(W) < 0 \).

We define the measure of absolute risk aversion as ARA.

\[ \text{ARA} = -\frac{U''(W)}{U'(W)} \]

It is called absolute risk aversion because it measures risk aversion for a given level of wealth. It provides much insight into people’s behaviour in the face of risk. For example, how does ARA change with one’s wealth level? Casual observations tell us that ARA will probably decrease as our wealth increases. A $1000 gamble may seem trivial to a billionaire, but an individual with limited total wealth, may be very risk averse to it. Another common measure is RRA, which is simply the ARA multiplied by the wealth level.

\[ \text{RRA} = -W \frac{U''(W)}{U'(W)} \]

A constant relative risk aversion implies that an individual will have constant risk aversion to a proportional loss of wealth even though the absolute loss increases as wealth does.

**Example:** Quantify the ARA and RRA of the utility function \( U(W) = -W^{-1} \).

\( U(W) = -W^{-1}: U'(W) = W^{-2}; U''(W) = -2W^{-3} \): This leads to

\[ \text{ARA} = -\frac{-2W^{-3}}{W^{-2}} = \frac{2}{W} \]

\[ \text{RRA} = 2 \]