Futures replication and the Law of One Futures Price

Avi Bick
Beedie School of Business
Simon Fraser University
Burnaby, B. C.  V5A 1S6, Canada
Tel. 778-782-3748
E-mail: bick@sfu.ca

Last Revised: March 2016

ABSTRACT
The paper analyzes zero-value trading strategies, which may be regarded as synthetic futures contracts. An “actual” futures contract may then be replicated by such a strategy, in the sense of matching cumulative cash flows. The Law of One Futures Price justifies futures pricing based on such a comparison.

Key words: trading strategies, futures pricing, futures-on-futures, derivatives replication, Law of One Futures Price.

*Previous versions of this paper were presented at the 2005 Northern Finance Association meetings, at the 2006 China International Conference in Finance and at the 2006 4th World Congress of the Bachelier Finance Society. The author is grateful for helpful suggestions obtained from workshop participants, and especially for comments by Sergey Isaenko and Jin Zhang. Discussions with Haim Reisman also helped to improve the paper.
1 Introduction

In their seminal paper, Harrison and Pliska (1981) noted that the dynamic replication of a synthetic payoff is of interest in its own right, and the pricing-by-arbitrage implication, if such an option is already available as a traded security, is just a simple second step. In this paper, analogous work is done for futures. Futures contracts are zero-value (ex-payouts) securities which produce a cash flow stream (“marking-to-market”) based on the increments in the futures price. It is natural to attempt to replicate such a security by a zero-value trading strategy which matches the terminal cumulative cash flow. This is especially appealing for “futures-on-futures” because a trading strategy in the underlying futures contracts is automatically zero-value. The second step is the assertion that the futures price of an “actual” contract with a given terminal condition must be equal to the cumulative cash flow (plus a constant) generated by the replicating strategy. This is justified by the Law of One Futures Price.

Thus, instead of employing self-financing trading strategies with bulk terminal payoffs, we use zero-value trading strategies with a continuous cash flow stream. True, a strategy of the second type can be transformed into a strategy of the first type, and our approach is definitely not essential for futures pricing. However, in situations where the underlying is a futures price vector, futures replication is simpler (requiring only one security, in the simplest case) and it is all that we need for futures pricing. Futures replication can also be applied in the case of futures on a spot price.

The Law of One Futures price was applied (with different terminology) in Jamshidian (1994), where it is used as a tool, and it is not the main focus of the paper. Several futures pricing results, for very specific contracts and under restricted conditions on the underlying stochastic processes, are derived via futures replication. In contrast, in the current paper we will explicitly formulate a “futures replication problem”, and it will be solved in three different settings for rather general terminal conditions. In analogy to the Harrison-Pliska contribution, the pricing end results are known, but the novelty is in the approach. Creating synthetic futures contracts is interesting in its own right, regardless whether an “actual” futures contract exists as a tradable security.

The paper is organized as follows: The setting is outlined in Section 2. In section 3, futures contracts are defined and The Law of One Futures Price is formulated. Section 4 includes three examples where a “futures replication problem” is solved. Section 5 is a summary.
2 The setting

A frictionless financial market is open for trade over the time interval \([0, T]\). Cash flows and values are measured in some units of account. Uncertainty is represented by a filtered probability space where the filtration \(\mathcal{F} = \{\mathcal{F}_t; t \in [0, T]\}\) represents the flow of information. We limit ourselves to the case where it is generated by a (vector) Brownian motion.

A set of \(m\) “reference” securities, which are characterized by two adapted (column) \(m\)-dimensional stochastic processes: \(\{D(t); t \in [0, T]\}\), where \(D(0) = 0\), represents cumulative payouts (possibly “dividends” or “marking-to-market” settlements), whereas \(\{S(t); t \in [0, T]\}\) represents ex-payout prices. Informally, \(dD_j(t)\) and \(dS_j(t)\) are the payout and price appreciation, respectively, of Security \(j\) over \([t, t + dt]\). We do not rule out negative payouts.

One can then define a trading strategy as a (row) \(m\)-dimensional adapted stochastic process \(\phi = \{\phi(t); t \in [0, T]\}\) representing the number of units held “immediately after the revision” at time \(t\). (We follow the convention as in Musiela and Rutkowski (2005), Chapter 2.) The interpretation is that the portfolio is revised at ex-payout prices. We then define the time-\(t\) value of the strategy “immediately after the time-\(t\) revision” as the scalar product \(V(t; \phi) = \phi(t)S(t)\) for \(t \in [0, T]\). The process \(\{D(t; \phi); t \in [0, T]\}\) representing the strategy's cumulative cash flow (immediately after the revision) is defined via \(D(0; \phi) = 0\) and

\[
dD(t; \phi) = \phi(t) dD(t) - d\phi(t)(S(t) + dS(t)).
\]

Again, on the RHS we use scalar products. The equation says that the incremental cash inflow over \([t, t + dt]\) is equal to the payouts minus the cost of the portfolio's revision at time \(t + dt\).

The formal interpretation is in the context of stochastic integration. See, e.g., the M-R book for definitions and technical conditions. Our treatment is less formal, for example like in Back (2005).

We say that \(\phi\) is a “zero-value (ex-payouts) strategy” if \(V(t; \phi) = 0\) with probability 1 for each \(t \in [0, T]\). We call \(\phi\) “self-financing” if, for each \(t\), \(dD(t; \phi) = 0\), namely if the time-\(t\) cumulative payout process is zero with probability 1.

It is assumed that one of the given securities is the (reinvested) money market fund, defined as a non-dividend-paying security whose price \(M(t)\) satisfies \(M(0) = 1\) and \(dM(t) = M(t) r(t) dt\), where \(r\) is an Ito process interpreted as the short-term rate. Then the following lemma is a variant of Proposition 2 in Cox, Ingersoll and Ross (1981). The reader
may regard “Security 1”, specified below, as a futures contract. At this point we only need the zero-value property.

**LEMMA 2.1:** Converting a zero-value security into a self-financing strategy.

Suppose that one of the given securities (“Security 1”, just for concreteness) has zero value all the time. Its cumulative cash flow \{D_1(t); t \in [0,T]\} is assumed to be an Ito process. A money market fund as above is also available. Now consider a trading strategy \(\varphi\) in these two securities, where \(\varphi^1(t) = M(t)\), \(\varphi^M(t) = D_1(t)\). Then \(\varphi\) is self-financing. If, in addition, \(D_1(T)\) is deterministic, no-arbitrage entails that \(D_1(t) = D_1(T)\) for each \(t\).

**Proof:** For our two-security strategy, Eq. (1) becomes (with abbreviated notation)

\[
dD(t; \varphi) = M dD_1 + D_1 \cdot 0 - dM \cdot 0 - dD_1(M + dM) = 0. \tag{2}
\]

This establishes self financing. The time-\(t\) position value is obviously \(D_1(t)M(t)\), and in particular it is \(D_1(T)M(T)\) at time \(T\). Now if \(D_1(T)\) is deterministic, the payoff \(D_1(T)M(T)\) can also be obtained by holding \(D_1(T)\) units of the money market fund. No arbitrage entails that \(D_1(t)M(t) = D_1(T)M(t)\) for each \(t\).  

### 3 Futures contracts

In principle, a derivative security is defined via the terms in the contractual agreement, as it would be specified in a legal document. For a futures contract in a discrete-time theoretical framework, with a terminal futures price which is a given function of some state variables, such a definition needs to be recursive. At each settlement time the payment is defined as the increment in the futures price (“proceeds from marking-to-market”), which, in turn, is determined in the market such that there is no need to pay in order to assume a position. (See Duffie (2001), p. 42 and Vellekoop (2010).)

However, for our purposes, especially in a continuous-time setting, it is more convenient to regard being a futures contract as a *property* of a security or a trading strategy. Our definition below is motivated by Duffie and Stanton (1992), Jamshidian (1994), Carr and Jarrow (1995), Duffie (2001), Section 8.C, Aase (2002) and Pozdnyakov and Steele (2004). Like in these works, the role of the exchange and a margin account are not part of the formalism.

Suppose \(\phi\) is a trading strategy in the reference securities over \([0,T]\) and \(X\) is an \(\mathcal{F}_T\)-measurable random variable. The pair \((\phi, X)\) is a “\(T\)-expiration futures contract” (with terminal condition \(X\)) if (i) \(\phi\) is a zero-value (ex-payouts) trading strategy, and (ii) there
exists a constant $F(0)$ such that, from time-0 perspective, with probability 1,

$$X = F(0) + D(T; \phi).$$

(3)

For better clarity, if $\phi$ is nontrivial (that is, it is not the buy-and-hold position in a single reference security) we may call $(\phi, X)$ a “synthetic futures contract”. However, we allow the possibility that $\phi$ is trivial, that is, a given reference security may be a futures contract.

For a (possibly synthetic) futures contract $(\phi, X)$, for each $t \in [0, T]$, the time-$t$ associated futures price is defined as

$$F(t; T, \phi, X) = X - D(T; \phi) + D(t; \phi) = F(0; T, \phi, X) + D(t; \phi),$$

(4)

where $F(0; T, \phi, X) = F(0)$ from (3). In what follows, $F(t) = F(t; T, \phi, X)$. Eq. (4) gives

$$dD(t; \phi) = dF(t), \quad F(T) = X.$$  

(5)

We may regard (5) as specifying a futures replication problem: For a given “boundary condition” $X$, find a zero-value trading strategy $\phi$ over $[0, T]$ and an adapted process $\{F(t); t \in [0, T]\}$ such that (5) (and hence (3) and (4)) are satisfied.

We note that $\phi$ by itself defines a terminal condition and a futures price process only to within an additive constant. See “Example” in Section 3. In general, if $(\phi, X)$ is a futures contract and $k$ is a constant, then $(\phi, X + k)$ is also a futures contract, and the relation between the associated futures prices is $F(t; T, \phi, X + k) = k + F(t; T, \phi, X)$.

We conclude the section with the analogue of the Law of One Price. It is essentially a restatement of Theorem 3.1 in Jamshidian (1994). The assertion is not self-evident because futures prices are cumulative cash flows.

**PROPOSITION 3.1:** The “Law of One Futures Price”.

If there is no arbitrage, if two (possibly synthetic) futures contracts have the same terminal condition, then the associated futures prices must be the same at any time.

**Proof** (outline): Apply Lemma 2.1 where “Security 1” is the position of buying one contract and selling the other. ■

### 4 Three examples of futures replication

With a given futures contract regarded as the underlying, one may consider “futures on futures” contracts. That is, the contract's time-$T$ terminal futures price is of the form $\tilde{g}(F(T))$, where $\tilde{g}$ is a specified function and $F(T)$ is the time-$T$ underlying futures price.
(The underlying contract may expire at $T^* \geq T$.) Such a contract, especially with a call-like or a put-like boundary condition, is sometimes called a “pure futures option” (Duffie (1989)) or an “option with futures price margining” (Lieu (1990)).

From a futures pricing perspective, the result in Proposition 4.1 below already appears in Lieu's paper. He uses “classical” (hedging-based) methods to derive the PDE, and then the solution for the boundary condition $(F - K)^+$ is a Black-Scholes-like formula. Our focus is on the futures replication problem, specifying a single-security zero-value trading strategy $\phi$ which matches (in the sense of (3)) a terminal futures price $X = \bar{g}(F(T))$ as above.

**PROPOSITION 4.1:**

(i) Suppose $\{F(t); t \in [0, T]\}$ is the futures price process associated with a given “reference” futures contract. It is assumed to be a positive Ito process satisfying

$$
(dF(t)/F(t))^2 = \sigma^2(F(t), t) \ dt,
$$

where $\sigma(x, t)$ is a suitably well-behaved continuous function.

(ii) Using subscripts to denote partial derivatives, suppose $g(x, t)$ is a suitably smooth function in $(x, t) \in \mathbb{R}^+ \times [0, T]$ which is the solution of the PDE

$$
g_t + \frac{1}{2} \sigma^2(x, t)x^2 g_{xx} = 0, \quad g(x, T) = \bar{g}(x),
$$

where $\bar{g} : \mathbb{R}^+ \to \mathbb{R}$ is a given continuous function.

Now consider the zero-value trading strategy $\phi(t) = g_x(F(t), t)$ in the reference futures contract. Then the incremental cash flow $dD(t; \phi)$ satisfies

$$
dD(t; \phi) = g_x(F(t), t) dF(t) = dg(F(t), t).
$$

In words: $(\phi, \bar{g}(F(T)))$ is a (synthetic) futures contract whose time-$t$ associated futures price is $g(F(t), t)$.

**Proof:** The first equality in (8) is based on Eq. (1). The second equality follows from Ito's lemma, combined with (6) and (7).

In analogy to Harrison and Pliska (1981), futures pricing is a by-product of the replication: Under the above assumptions, suppose that one of the traded securities is another $T$-expiration futures contract with an associated futures price process $\{H(t)\}$ such that $H(T) = \bar{g}(F(T))$. The Law of One Futures Price says that $H(t) = g(F(t), t)$ for $t \leq T$. 

-5-
**Example:** Suppose \( \sigma \) is a constant. Consider the PDE (7) with \( \overline{g}(x) = x^2 + k \), where \( k \) is a constant. Then the solution is \( g(x,t) = a(t)x^2 + k \), where \( a(t) = \exp(\sigma^2(T-t)) \). For a given underlying futures contract with associated futures price \( F \), suppose that the dynamics of \( F \) is as in Eq. (6). Consider the zero-value *single-security* trading strategy in this contract defined by \( \phi(t) = 2a(t)F(t) \). It does not depend on \( k \). The proposition says that \( \phi \) generates the cumulative cash flow \( (F(T))^2 - a(t)(F(t))^2 \) over \( [t,T] \). Thus, for any \( k \), \( \left( \phi, (F(T))^2 + k \right) \) is a synthetic futures contract and the associated time-\( t \) futures price is \( a(t)(F(t))^2 + k \). If there already exists a futures contract whose futures price process is \( \{H(t)\} \) so that \( H(T) = (F(T))^2 + k \) for some \( k \), then the Law of One Futures Price says that we should have \( H(t) = a(t)(F(t))^2 + k \) for \( t \leq T \).

A generalized version of Proposition 4.1, where \( \{F(t)\} \) is \( n \)-dimensional is available in a previous version of this paper. The same is true for Proposition 4.2 below.

In the next example, the zero-value trading strategy depends on \( F_1/F_2 \), where \( F_1 \) and \( F_2 \) are futures prices. The terminal condition is of the form \( F_2 \cdot \overline{h}(F_1/F_2) \), for example \( \min(k_1F_1, k_2F_2) \) or \( \max(F_1-F_2, 0) \). (Futures pricing for the former is discussed in Bick (1977).)

**PROPOSITION 4.2:**

(i) Suppose \( \{F(t); t \in [0,T]\} \) is a vector Ito process of two futures prices over \( [0,T] \), and suppose \( F_2(t) > 0 \) (a.s.) for each \( t \). Let \( Y(t) := F_1(t)/F_2(t) \). Suppose it satisfies

\[
(dY(t)/Y(t))^2 = \sigma^2(Y(t),t) \, dt,
\]

where \( \sigma(y,t) \) is a suitably well-behaved continuous function.

(ii) Suppose \( h : \mathbb{R} \times [0,T] \rightarrow \mathbb{R} \) is the solution of

\[
h_t(y,t) + \frac{1}{2} \sigma^2(y,t) \, y^2 h_{yy}(y,t) = 0, \quad h(y,T) = \overline{h}(y),
\]

where \( \overline{h} : \mathbb{R} \rightarrow \mathbb{R} \) is a specified continuous function.

Now consider the zero-value trading strategy \( \phi \) in the two futures contracts, where

\[
\phi_1(t) = h_y(Y(t),t), \quad \phi_2(t) = h(Y(t),t) - \phi_1(t) \, Y(t).
\]

Then the incremental cash flow \( dD(t;\phi) \) satisfies

\[
dD(t;\phi) = d(F_2(t)h(Y(t),t)).
\]
Thus we may say that \((\phi, F_2(T)\bar{h}(Y(T)))\) is a (synthetic) \(T\)-expiration futures contract whose associated time-\(t\) futures price is \(F_2(t) h(Y(t), t)\).

**Proof** (outline): Based on Eq. (1), Eq. (12) can be written (in abbreviated notation) as

\[
h_y(Y, t) dF_1 + [h(Y, t) - h_y(Y, t) Y] dF_2 = d(F_2 h(Y, t)).
\] (13)

To prove (13), apply the product rule to the RHS of (13) and also to \(dF_1 = d(F_2Y)\). Also use the equality \(dh(Y, t) = h_y(Y, t) Y \ dY\), which follows from Ito's lemma and Eqs. (9)-(10). □

The futures pricing implication is analogous to the one based on Proposition 4.1.

Finally, let us apply our approach in the case of futures on a spot price. In what follows, an alternative derivation will be provided to a futures pricing result (Eq. (4)) from Brennan and Schwartz (1985). We will “synthesize” the futures contract with a zero-value trading strategy in the spot asset and the money market fund (defined in Section 2).

**PROPOSITION 4.3:**

Suppose a stock price follows a positive Ito process \(\{S(t); t \in [0, T]\}\) such that

\[(dS(t)/S(t))^2 = \sigma^2(S(t), t) \ dt.\] (14)

Suppose the stock pays a dividend stream at a rate \(\delta(S(t), t)\), and the instantaneous interest rate is also of the form \(r(S(t), t)\) (or deterministic \(r(t)\), as a special case). For a given continuous function \(\bar{g} : \mathbb{R}^+ \rightarrow \mathbb{R}\), suppose \(g : \mathbb{R}^+ \times [0, T] \rightarrow \mathbb{R}\) is the solution of

\[
g_t(s, t) + \frac{1}{2} \sigma^2(s, t) s^2 g_{ss}(s, t) + g_s(s, t) \cdot (r(s, t) - \delta(s)) = 0,
\] (15)

\[g(s, T) = \bar{g}(s).\] (16)

Consider a zero-value trading strategy \(\phi\) in the stock and the money market fund where

\[\phi^S(t) = g_s(S(t), t), \quad \phi^M(t) = -\phi^S(t) S(t)/M(t).\] (17)

Then its incremental cash flow is

\[dD(t; \phi) = dg(S(t), t).\] (18)

In other words, \((\phi, \bar{g}(S(T)))\) is a (synthetic) \(T\)-expiration futures contract whose time-\(t\) associated futures price is \(g(S(t), t)\).
**Proof** (outline): Eq. (1) becomes (in abbreviated notation)

\[
dD(t; \phi) = \phi^S \cdot (dS + \delta S \, dt) + \phi^M \cdot (Mr \, dt) \\
= gs(S, t) \cdot (dS + \delta S \, dt - S \, r \, dt) = dg(S, t).
\]

(19)

The third equality follows from Ito's lemma and Eqs. (14)-(15).

**5 Summary**

The paper formulates and solves (in three examples) a “futures replication problem”. The link to futures pricing is via the Law of One Futures Price.

**References**


