Problem Set #1 Answer Key

Economics 808: Macroeconomic Theory

Fall 2004

1 The Cobb-Douglas production function

a) First we show that it is homogeneous of degree 1:

$$\begin{aligned} A(cK)^{\alpha}(cL)^{1-\alpha} &= Ac^{\alpha}K^{\alpha}c^{1-\alpha}L^{1-\alpha} \\ &= cAK^{\alpha}L^{1-\alpha} \end{aligned}$$

Next we show that both factors are necessary:

$$\begin{aligned} A(0)^{\alpha}L^{1-\alpha} &= 0 * AL^{1-\alpha} = 0 \\ AK^{\alpha}(0)^{1-\alpha} &= 0 * AK^{\alpha} = 0 \end{aligned}$$

Next, that both factors contribute to output:

$$\begin{aligned} \frac{\partial F}{\partial K} &= \alpha AK^{\alpha-1}L^{1-\alpha} \\ \frac{\partial F}{\partial L} &= (1-\alpha)AK^{\alpha}L^{-\alpha} \end{aligned}$$

Since $0 < \alpha < 1$, each of these is a string of strictly positive numbers being multiplied together. So each is positive.

Next, we show that F is concave

$$\frac{\partial^2 F}{\partial K^2} = \alpha(\alpha-1)AK^{\alpha-2}L^{1-\alpha}$$

All of these factors are positive with the exception of $\alpha - 1$. A negative number multiplied by a positive number produces a negative number, so this is negative.

$$\frac{\partial^2 F}{\partial L^2} = (1-\alpha)(-\alpha)AK^{\alpha}L^{-\alpha-1}$$

All of these factors are positive with the exception of $-\alpha$. So this quantity is negative.

Next we show that the Inada conditions hold. First we rearrange:

$$\frac{\partial F}{\partial K} = \frac{\alpha AL^{1-\alpha}}{K^{1-\alpha}}$$

when L is positive and finite, the numerator is also positive and finite. When $K \to 0$, the denominator also approaches zero, so the entire expression is approaching infinity (Inada condition #1). When $K \to \infty$, the denominator approaches infinity, so the entire expression approaches zero (Inada condition #2).

So the Cobb-Douglas production function satisfies all neoclassical assumptions.

b) Let $\hat{A} = A^{1/(1-\alpha)}$. Then $Y = K^\alpha (\hat{A}L)^{1-\alpha}$.

c)

$$\begin{aligned} w_t &= \frac{\partial F}{\partial L_t} \\ &= (1-\alpha)AK_t^\alpha L_t^{-\alpha} \\ &= (1-\alpha)A\left(\frac{K_t}{L_t}\right)^\alpha \end{aligned}$$

d)

$$\begin{aligned} r_t &= \frac{\partial F}{\partial K_t} \\ &= \alpha AK_t^{\alpha-1} L_t^{1-\alpha} \\ &= \alpha A\left(\frac{L_t}{K_t}\right)^{1-\alpha} \end{aligned}$$

e) This model implies that wages are increasing in the capital/labor ratio and that interest rates are decreasing in that ratio. Therefore, we should expect to see low wages and high interest rates in poorer countries.

f)

$$\begin{aligned} \frac{w_t L_t}{Y_t} &= \frac{\left(1-\alpha)AK_t^\alpha L_t^{-\alpha}\right) L_t}{AK_t^\alpha L_t^{1-\alpha}} \\ &= 1-\alpha \end{aligned}$$

g)

$$\begin{aligned} \frac{r_t K_T}{Y_t} &= \frac{\left(\alpha AK_t^{\alpha-1} L_t^{1-\alpha}\right) K_t}{AK_t^\alpha L_t^{1-\alpha}} \\ &= \alpha \end{aligned}$$

2 Constant and increasing returns in the Solow model

a) If $a < w$

1. Profit-maximizing output is zero

2. Profit-maximizing labor demand is zero
3. Total profit is zero

If $a = w$

1. Any level of output is profit-maximizing.
2. Any level of labor demand is profit-maximizing.
3. Total profit is zero

If $a > w$

1. Profit-maximizing output is infinity. It's actually not right to say this, since infinity isn't a real number. A better way of putting this is that there does not exist a profit-maximizing level of output, because any arbitrarily large amount of profit can be made. The same language applies to labor demand and total output.
2. Profit-maximizing labor demand is infinity.
3. Total profit is infinity.

b) As long as there is a positive supply of labor, markets will clear only if firms demand that level of labor. The market-clearing wage is a. At that wage, firms are indifferent as to the amount of output they sell and the amount of labor they buy. Firm profits at that wage are zero.

c) For any (finite) wage, profit maximizing output, labor demand, and profit will be infinity.

How could you prove this? Use a standard proof by contradiction:

Proposition: There does not exist $L \in [0, \infty)$ which maximizes $aL^2 - wL$.

Proof: Suppose not, that is, suppose that there exists some ℓ that maximizes profits. Well, $a(\ell + 1)^2 - w(\ell + 1) > a\ell^2 - w\ell$, so ℓ cannot maximize profits.

So there is no market clearing wage.

3 More on constant returns

Profits for a firm are:

$$\pi = F(K, AL) - rK - wL$$

which we can rewrite as:

$$\pi = L[F(K/L, A) - rK/L - w]$$

This can now be restated in terms of a maximization problem in which each firm chooses L and K/L to maximize profits.

Since L cannot be negative, maximizing the above expression implies maximizing $[F(K/L, A) - rK/L - w]$. Since there is a unique value of K/L which maximizes that expression for a given set of prices, that means all firms will choose the same K/L, no matter how big or small.

Notice that this has several implications. First, you can view the choices of firms in this model as having two components - a very important cost-minimization problem in which they attempt to get the most efficient combination of capital and labor, and a (irrelevant in equilibrium because of zero profits) "scale of operation" problem. Second, prices are are determined by the capital/labor ratio. Third, our analysis in the previous problem (which ignores capital) carries over easily to the case here.

4 Growth accounting

a) I chose to analyze Canada's history from 1965 to 1990. The results are in the Excel file solow.xls on the class web site.

b)

1. My estimates imply that, while there is a long-run trend of growth in TFP, annual productivity growth is quite unpredictable.

2. My estimates show negative productivity growth in 7 out of the 25 years in the sample. If we interpret TFP as representing the state of technology, this implies that Canada experienced technological regress in those years.

3. There is a popular school of thought in macroeconomics, the "Real Business Cycle" school, that believes Solow residuals are good measures of technological progress both in the short run and the long run. In their view, recessions and booms are driven by random fluctuations in the rate at which new technologies appear. A different school of thought (which includes Robert Solow himself) agree with the RBC school that the Solow residual measures productivity growth reasonably well, and that in the long run productivity growth is associated with technological progress. But they believe that short-run fluctuations in measured productivity can be traced to a variety of causes including technological progress, misallocation of resources during a recession, cyclical utilization (firms may have the same amount of capital, but use it less intensively during recessions), etc. I tend to agree with the second school of thought, and the most compelling evidence to me is that the RBC interpretation implies that episodes of large-scale technological regress happen during deep recessions.