Problem Set #2

Economics 808: Macroeconomic Theory

Fall 2004

1 Linear difference equations

Modern macro is fundamentally dynamic - the output of a macro model is usually a system of *difference equations* - equations relating the state of the economy at time t to the state at time $t+1$. One of the best ways to understand the dynamics of a model is to simulate it on a computer and play with it. For this problem you will need a computer. The problem can be done in a spreadsheet such as Microsoft Excel, but you can use any program you wish.

First we consider a linear difference equation, which is simply an equation of the form:

$$x_{t+1} = ax_t$$

We are going to simulate this difference equation for various values of a.

a) Let $a = -1.5$. Draw (using your computer program) a random initial condition x_0. Use the uniform distribution on the interval $[0, 1]$. Then use the difference equation to generate the next 49 values of x_t, so that you have a list of 50 values. Create a plot of the time series x_t. All I want is the plot (no other computer printouts needed), but the plot should be labeled clearly. Make sure that the plot is representative of the system's typical behavior by running it for a few different initial conditions (you only need to turn in one of these plots).

b) Repeat for $a = -0.5$.

c) Repeat for $a = 0.5$.

d) Repeat for $a = 1.5$.

2 A nonlinear difference equation

Some nonlinear systems behave very similarly to linear systems, while others produce very complex dynamics. To see the range of behavior, consider the difference equation $x_{t+1} = f(x_t)$ for $f(x) = 0.5x^{0.5}$ $(x \geq 0)$.

a) Plot $y = f(x)$ for the range $x \in [0, 1]$ on a graph. Also plot the line $y = x$ on the same graph. Use the computer to generate the graph.

b) What are the fixed points of f? A fixed point is simply an x that solves the equation $x = f(x)$.

c) What is the value of $f'(x)$ at each fixed point, and what does that imply about the behavior of the system near the fixed point?

d) Starting with an initial condition drawn from the uniform distribution on $[0, 1]$, simulate and plot a time series (50 periods) as in the previous exercise.

3 Another nonlinear difference equation

Next consider $f(x) = ax(1 - x)$.

a) Let $a = 1.1$. Plot $f(x)$ and $y = x$.

b) What are the fixed points of f?

c) What is the value of $f'(x)$ at each fixed point, and what does that imply about the behavior of the system near the fixed point?

d) Simulate and plot a time series as in the previous exercise.

e) Let $a = 2.9$. Plot $f(x)$ and $y = x$.

f) What are the fixed points of f?

g) What is the value of $f'(x)$ at each fixed point, and what does that imply about the behavior of the system near the fixed point?

h) Simulate and plot a time series as in the previous exercise.

i) Let $a = 3.97$. Plot $f(x)$ and $y = x$.

j) What are the fixed points of f?

k) What is the value of $f'(x)$ at each fixed point, and what does that imply about the behavior of the system near the fixed point?

l) Simulate and plot a time series as in the previous exercise. This is an example of a chaotic difference equation - the time series should look like a fairly random sequence of numbers.

4 The CES production function

Suppose the world is described by Solow model and that the production function has the constant elasticity of substitution (CES) production form:

$$F(K, L) = [aK^\rho + bL^\rho]^{1/\rho}$$

Assume $g_L = g_A = 0$, and that $A_t = L_t = 1$ for all t.

a) Find the intensive form of this production function $f(k_t)$

b) Find the rental rate on capital r_t in terms of k_t

c) Find the wage w_t in terms of k_t

d) Find the growth rate of capital in terms of current capital k_t and model parameters.

e) Find the steady-state value of k_t in terms of model parameters.

5 Capital flows in the Solow model

Suppose that there are two countries, Scotland and Canada. Both countries are well-described by the Solow model and have identical Cobb-Douglas production functions:

$$Y = AK^{\alpha}L^{1-\alpha}$$

The countries have the same level of technology (A) but have different capital to labor ratios, and thus different per capita income. Define the gross return on capital as $r_t + 1 - \delta$ (in other words, the rental value for the current period, plus the amount of undepreciated capital left at the end of the period. Define the net return on capital as $r_t - \delta$, or the gross return minus one. Let $\delta = 0$, and $\alpha = \frac{1}{3}$.

a) If Canada has 10 times Scotland's per capita income, what is the ratio of per-capita capital in Canada to that in Scotland?

b) If Canada has a 5 percent net return on capital, (and 10 times the per capita income of Scotland), what is the net return on capital in Scotland?

c) If you were a Canadian under these circumstances with some savings to invest, would you invest it in Canada or or send it to Scotland?