Problem Set #2 Answer Key

Economics 808: Macroeconomic Theory

Fall 2004

1 Linear difference equations

See the Excel file diffeq.xls on the web site.

2 A nonlinear difference equation

a) See the computer file.

b) We find a fixed point by solving the equation $x_\infty = 0.5x_\infty^{0.5}$ for x_∞. This gives us two fixed points, $x_\infty \in \{0, 0.25\}$.

c) In general, $f'(x) = \frac{0.25}{x^{0.5}}$, so $f'(0) = \infty$[1] and $f'(0.25) = 0.5$. Zero is an unstable fixed point, meaning that if the system starts near zero, it will tend to move away from zero. In contrast, the 0.25 is a stable fixed point, implying that if the system starts near 0.25 it will tend to move towards 0.25. In addition, because $f'(0.25) > 0$, it will move monotonically towards 0.25.

d) The simulated time series (in the computer file) shows x_t monotonically converging to 0.25, just as the local analysis predicted.

3 Another nonlinear difference equation

a) See the computer file.

b) We find a fixed point by solving the equation $x_\infty = ax_\infty(1 - x_\infty)$ for x_∞. In general there are two solutions:

$$x_\infty \in \{0, \frac{a-1}{a}\}$$

When $a = 1.1$, the two fixed points are 0 and 0.09.

c) In general $f'(x) = a(1 - 2x)$. This means

$$\begin{aligned} f'(0) &= a \\ f'(\frac{a-1}{a}) &= 2 - a \end{aligned}$$

[1] If you prefer, $\lim_{x\to 0} f'(x) = \infty$

In this case, $f'(0) = 1.1$ and $f'(0.09) = 0.9$. Zero is an unstable fixed point and 0.09 is a stable fixed point.

d) The simulated time series (in the computer file) shows that x_t converges monotonically to the stable fixed point.

e) See the computer file.

f) The function f has two fixed points, 0 and 0.655.

g) In this case $f'(0) = 2.9$ and $f'(0.655) = -0.9$. Zero is an unstable fixed point, and the 0.655 is stable. Since $f'(0.655) < 0$, this implies that starting near 0.655, the system will converge to 0.655 with oscillations.

h) The simulated time series (in the computer file) shows that x_t moves towards the stable fixed point, but oscillates, as predicted by the local analysis.

i) This plot is in the computer file.

j) The fixed points of f are zero and 0.748.

k) In this case $f'(0) = 3.97$ and $f'(0.748) = -1.97$. Both fixed points are unstable, implying that starting near either, the system will move away.

l) The simulated time series shows that x_t appears to jump around randomly with no clear pattern even though each value is a deterministic function of the previous value.

The time series you see here is an example of a *chaotic* system. Chaos only appears in systems governed by deeply nonlinear difference equations. In fact, the way you get "random" numbers from a computer is through successive application of a nonlinear difference equation to some "seed" value (say, the current date in seconds since Jan 1, 1970).

This particular difference equation exhibits chaos when $3.94 < a < 4$.

4 The CES production function

The purpose of this problem is mostly to give you practice in working with these things under a production function other than the Cobb-Douglas one. As you can see, the CES production function give much messier results than Cobb-Douglas. This problem is quite similar to Romer's Problem 1.3.

a) The intensive form is $f(k) = F(k, 1)$, so

$$f(k) = [ak^\rho + b]^{1/\rho}$$

b) We know that $r_t = f'(k_t)$, so:

$$r_t = ak_t^{\rho-1}\left[a + bk_t^{-\rho}\right]^{\frac{1-\rho}{\rho}}$$

c) Since $L_t = 1$:

$$w_t = b\left[ak_t^\rho + b\right]^{\frac{1-\rho}{\rho}}$$

d) The basic capital accumulation equation is:

$$k_{t+1} = (1-\delta)k_t + s[ak_t^\rho + b]^{1/\rho}$$

Subtracting k_t and then dividing by k_t, we get:

$$\frac{k_{t+1} - k_t}{k_t} = s[a + bk_t^{-\rho}]^{1/\rho} - \delta$$

You might notice that this growth rate is decreasing in k_t.

e) k_t is at the steady state when the growth rate we found above is equal to zero:

$$s[a + bk_\infty^{-\rho}]^{1/\rho} = \delta$$

Solving, we get:

$$k_\infty = \left(\frac{b}{(\delta/s)^\rho - a}\right)^{1/\rho}$$

5 Capital flows in the Solow model

a) First we note that:

$$\frac{y_C}{y_S} = \frac{Ak_C^\alpha}{Ak_S^\alpha}$$

and we solve for $\frac{k_C}{k_S}$ as a function of $\frac{y_C}{y_S}$ and model parameters. This gives

$$\begin{aligned}\frac{k_C}{k_S} &= \left(\frac{y_C}{y_S}\right)^{1/\alpha} \\ &= 10^3 \\ &= 1000\end{aligned}$$

In other words, in order to produce 10 times Scotland's output, Canada must have 1000 times Scotland's capital.

b) We know from the firm's profit maximization problem that

$$r = f'(k) = \alpha Ak^{\alpha-1}$$

So

$$\begin{aligned}r_S &= \alpha Ak_S^{\alpha-1} \\ &= \alpha A\left(\frac{k_S}{k_C}k_C\right)^{\alpha-1} \\ &= \left(\frac{k_S}{k_C}\right)^{\alpha-1} * \alpha Ak_C^{\alpha-1} \\ &= \left(\frac{k_C}{k_S}\right)^{1-\alpha} * r_C \\ &= 1000^{2/3} * 0.05 \\ &= 5.00\end{aligned}$$

So Scotland has a 500% net return on capital.

c) I don't know about you but I'd put my money in Scotland.

Of course, given that we see tenfold differences in output per worker, but don't see these kinds of returns in less-developed countries, something is going on here. Sure enough, poor countries have much lower TFP in addition to less capital.