Problem Set #4

Economics 808: Macroeconomic Theory

Fall 2004

1 The cake-eating problem

Consider the optimal growth problem (discrete time) where:

$$f(k) = k$$

This problem is commonly called a "cake-eating" problem. The consumer starts with a certain amount of capital, and "eats" it over time. We will use this problem to try some dynamic programming.

The planner's problem is to maximize:

$$\sum_{t=0}^{\infty} \beta^t \log c_t$$

subject to the constraints:

$$\begin{aligned} k_{t+1} &\leq k_t - c_t \\ k_t &\geq 0 \\ k_0 &> 0 \qquad \text{given} \end{aligned}$$

where log is the natural (base e) logarithm function.

a) Write down Bellman's equation for this problem.

b) First we will perform policy function iteration. First we guess at an optimal policy. I guess that the optimal policy is:

$$c_0(k) = (1 - \beta)k$$

Write down the value of k_t (for any $t > 0$) as a function of k_0, β, and t if this policy is followed.

c) Write down the value of c_t as a function of k_0, β, and t if this policy is followed.

d) Once you have a guess at the policy function, you calculate the value function under the assumption that this is the optimal policy. What is the value function if the equation above describes the optimal policy? (Just to keep things simple, feel free to drop any constant term).

e) The next step is to calculate the optimal policy under the new value function. This will give you a new policy function, which we will call $c_1(k)$. Find $c_1(k)$.

f) To find the true optimal policy, you keep applying these two steps until your policy function stops changing i.e., $c_i(k) = c_{i+1}(k)$. What is the true optimal policy function?

g) Next we will try value function iteration. First we guess at the form of the value function. Suppose that your initial guess for the value function is:

$$V_0(k) = \log k$$

Next, we calculate a new value function according to the formula:

$$V_{i+1}(k) = \max_{c \in [0,k]} \{\log c + \beta V_i(k-c)\}$$

Calculate V_1, V_2, and V_3. Feel free to throw out any constant terms. In other words, if

$$V_1(k) = \frac{\beta \log(\beta)}{\log(1+\beta)} + \beta \log k$$

then you can just write $V_1(k) = constant + \beta \log k$.

h) If you've done it right, you should see a pattern. Use this pattern to discern V_i for an arbitrary integer i.

i) What is the limit of this as $i \to \infty$?

j) What is the optimal policy function when this is the value function?

Note: In doing this problem it will be helpful to remember that for any number b such that $|b| < 1$, then:

$$\frac{1}{1-b} = \sum_{i=0}^{\infty} b^i$$

and that for any constant c:

$$\sum_{i=0}^{\infty} c x_i = c \sum_{i=0}^{\infty} x_i$$

2 Optimal growth with Cobb-Douglas utility

Consider the optimal growth model with Cobb-Douglas utility, Cobb-Douglas production, and 100% depreciation every period. The social planner maximizes:

$$\sum_{t=0}^{\infty} \beta^t \log c_t$$

subject to the constraints:

$$\begin{aligned} k_{t+1} &= A k_t^{\alpha} - c_t \\ k_t &\geq 0 \\ k_0 &> 0 \qquad \text{given} \end{aligned}$$

a) Write down the Bellman equation for this planner's problem.

b) Write down the necessary conditions for a solution to this planner's problem, including the transversality condition, and the Benveniste-Scheinkman formula for $V'(.)$

c) Derive the Euler equation.

d) Find the steady state values of k_t, c_t, and y_t.

e) Find the steady state savings rate.

f) Suppose that $\alpha = 1$, so that we have an "AK" model. Ignore for the time being the possibility that there is no solution and assume that the Euler equation applies. What is the growth rate of consumption?

3 Optimal growth with CRRA utility

Consider the optimal growth model with a constant relative risk aversion (CRRA) utility function, Cobb-Douglas production function, and 100% depreciation every period. The social planner maximizes:

$$\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

subject to the constraints:

$$\begin{aligned} k_{t+1} &= Ak_t^{\alpha} - c_t \\ k_t &\geq 0 \\ k_0 &> 0 \qquad \text{given} \end{aligned}$$

a) Write down the Bellman equation for this planner's problem.

b) Write down the necessary conditions for a solution to this planner's problem, including the transversality condition, and the Benveniste-Scheinkman formula for $V'(.)$.

c) Derive the Euler equation.

d) Find the steady state values of k_t, c_t, and y_t.

e) Find the steady state savings rate.

f) Suppose that $\alpha = 1$, so that we have an "AK" model. Ignore for the time being the possibility that there is no solution and assume that the Euler equation applies. What is the growth rate of consumption?

4 Optimal growth with linear utility

Consider the optimal growth model with linear utility, Cobb-Douglas production, and 100% depreciation every period. The social planner maximizes:

$$\sum_{t=0}^{\infty} \beta^t c_t$$

subject to the constraints:

$$\begin{aligned} k_{t+1} &= Ak_t^\alpha - c_t \\ c_t &\geq 0 \\ k_t &\geq 0 \\ k_0 &> 0 \qquad \text{given} \end{aligned}$$

I've added the additional constraint that consumption is positive. As always, assume $\alpha \in (0, 1)$, and $A > 0$.

a) Prove that there exists a solution to this optimal growth problem. You can do this by finding any finite number which is bigger than the highest possible utility level.

b) Describe as best you can the solution to this planner's problem. You should note that mechanical application of the first-order conditions and the Euler equation is unlikely to find the solution.

c) Suppose instead that $\alpha = 1$, so that we have an "AK" model. Identify conditions under which there is a solution to the planner's problem, and describe the solution under these conditions.