Problem Set #5 Answer Key

Economics 808: Macroeconomic Theory

Fall 2004

1 Human and physical capital in the RA model

a)

b) In order for consumer to accumulate both kinds of capital, it is necessary for $w_t = r_t$. This, combined with the standard profit maximization for the firm, gives:

$$\hat{h} = \frac{1-\alpha}{\alpha}$$

c) For time $t > 0$, this is:

$$w_t = r_t = A\alpha^{\alpha}(1-\alpha)^{1-\alpha}$$

d)

$$g_c = \ln \beta A + \alpha \ln \alpha + (1-\alpha)\ln(1-\alpha)$$

e) The time path or these three variables will be the same as before.

f) When $\frac{h_t}{k_t} < \hat{h}$, then $r_t < w_t$. So the consumer will invest in human capital but not in physical capital. This will continue until $\frac{h_t}{k_t} = \hat{h}$, at which point $r_t = w_t$ and the consumer will invest in both. From that point on, the two rates of return will be equal and $\frac{h_t}{k_t} = \hat{h}$ forever.

2 A small open economy

a) As there are numerous correct ways to set up the problem, I will not give you an answer here.

b) Ignore consumption for the moment and put yourself in the position of an investor. Investors always like opportunities to make money for nothing (otherwise known as arbitrage). So let's look for arbitrage opportunities.

Suppose you anticipate that $R_{t+1} > R$. In that case you could make arbitrarily large profits by borrowing at interest rate R, and lending at interest rate R_{t+1}. By similar reasoning, if you anticipate that $R_{t+1} < R$ you could make arbitrarily large profits by borrowing at R_{t+1} and lending at R. Since we have a market clearing condition that the total amount of local bonds purchased is zero, this implies that $R_{t+1} = R$ for all t. The same reasoning tells you that $r_{t+1} \leq R$, because otherwise investors could make arbitrarily large profits by borrowing at R and investing at r_{t+1}.

But the fact that investors cannot own negative amounts of physical capital means that we cannot use simple arbitrage to rule out $r_{t+1} < R$. Instead we note that if this were the case, no investor would purchase physical capital. But then $k_{t+1} = 0$, and since we have assumed the Inada conditions, this implies that r_{t+1} goes to infinity as k_{t+1} goes to zero. This contradicts our earlier assumption that $r_{t+1} < R$, so this cannot be an equilibrium. Therefore, the only equilibrium price is $r_{t+1} = R$, for all t.

What does this mean for the behavior of the capital stock? Well, this implies that k_{t+1} must solve $f'(k_{t+1}) = R$ for all t. Capital will immediately jump to a steady state level so that the returns to capital and the returns on the foreign bond market are identical. Notice that this steady state capital level has nothing to do with the agent's utility function (β doesn't appear here). Solving these things out for the Cobb-Douglas production function, we have:

$$\begin{aligned} r_{t+1} = R_{t+1} &= R \\ k_{t+1} &= \left(\frac{\alpha A}{R}\right)^{\frac{1}{1-\alpha}} \end{aligned}$$

The effect of the international bond market is to separate the worker's savings decision from his capital accumulation decision. If the worker currently has low capital, he will borrow enough money on the international bond market to buy the steady state level of capital immediately. In the future, any savings above and beyond that needed to maintain the capital stock will be used to purchase bonds (or to pay off bonds, which is really the same thing).

c) The Euler equation is:

$$u'(c_t) = \beta u'(c_{t+1}) r_{t+1}$$

Since we showed that $r_{t+1} = R$, this implies:

$$\frac{c_{t+1}}{c_t} = \beta R$$

So the growth rate of consumption is just:

$$g_c = \ln \beta R$$

d) The answer to the previous question implies that the growth rate of consumption is constant. So there is a positive-consumption steady state only if the growth rate is zero or if $R = \beta^{-1}$. If $R < \beta^{-1}$, consumption will decrease to zero over time. If $R > \beta^{-1}$ consumption will grow over time.

e) Consumption will reach its steady state value (if there is one) immediately.

3 Two economies

a) As there are numerous correct ways to set up the problem, I will not give you an answer here.

b) Steady-state capital is:

$$\begin{aligned} k_{A,\infty} &= (\alpha\beta_A)^{\frac{1}{1-\alpha}} \\ k_{B,\infty} &= (\alpha\beta_B)^{\frac{1}{1-\alpha}} \end{aligned}$$

c) Country A has more capital in the steady state.

d) As there are numerous correct ways to set up the problem, I will not give you an answer here.

e) The rental rate on capital is $r_t = 2^{1-\alpha}\alpha K_t^{\alpha-1}$

f) The Euler equations are now:

$$\begin{aligned} \frac{1}{c_{A,t}} &= \frac{\beta_A r_{t+1}}{c_{A,t+1}} \\ \frac{1}{c_{B,t}} &= \frac{\beta_B r_{t+1}}{c_{B,t+1}} \end{aligned}$$

If you apply the usual means of finding a steady state, you get the implication that $\beta_A = \beta_B$. This of course is false, so we have a problem.

How do we reconcile this? Let R, c_A, and c_b denote steady state interest rate, and consumption levels. In the steady state:

$$(\beta_A R - 1)c_A = 0$$
$$(\beta_B R - 1)c_B = 0$$

Since $\beta_A \neq \beta_B$, the only way these two equations can both be true is if c_A or c_B (or both) is zero. So any steady state is characterized by one of the countries having zero consumption.

What does this imply about the other variables? Suppose for a moment that $c_B = 0$ and $c_A > 0$. If this is true, then $\beta_A R = 1$, so $R = \frac{1}{\beta_A}$. This implies that

$$K_\infty = 2(\alpha\beta_A)^{\frac{1}{1-\alpha}}$$

Notice that this is exactly twice what country A's steady-state capital was before the economies were integrated.

The next question is how this capital is distributed among the two countries. The answer is "it doesn't matter". To see this, suppose that country A has 5 units of capital and country B has 1. Suppose country B offers country A a bond in exchange for one unit of capital. This gives A 4 units of capital and B 2 units. Both countries are indifferent between making this trade and not making this trade. Therefore, if one of these two distributions of capital is an equilibrium, so is the other.

However, the total value of capital and bonds held by each country *is* determined in the steady state. Country B has no income left for consumption. His steady-state budget constraint is:

$$c_B + b_B + k_B = Rb_B + Rk_B + w$$

In order for c_B to equal zero, it must be the case that: $b_B + k_B = -\frac{w}{R-1}$. In other words, country B owes country A so much money that his capital income (if there is any) and wage just barely cover interest on the debt.

The steady state in which $c_A = 0$ and $c_B > 0$ can be found by repeating the previous analysis with the countries switched, and the steady state in which no one consumes is the usual trivial steady state with no capital that you find in the basic RA model.

The next question is "which steady state does the economy converge to?" Your intuition should say that country B is more likely to accumulate a huge debt because they value the future less. That intuition is correct, but how to show it in the model?

Well, notice that the Euler equation implies that, at any point in time, the consumption growth rate in country A is bigger than the consumption growth rate in country B. That implies that if $c_{A,t}$ is going to zero in the long run, so is $c_{B,t}$. So the steady state in which country B consumes but country A does not is unstable - no economy will ever end up there from any other state. The same thing applies about the trivial steady state. So the steady state in which country A consumes but country B does not will be the one that is actually reached by this economy.

One last thing to notice. Despite the somewhat ghoulish implications (a country which is slightly less patient will eventually become slaves to a country that is more patient), the equilibrium here is Pareto efficient. In addition, the equilibrium allocation for the integrated economy Pareto dominates the equilibrium allocation without trade.

g) The two countries immediately trade bonds so that the $R_{t+1} = r_{A,t+1} = r_{B,t+1}$. The steady state in this economy is identical to the previous answer with one exception. Since the return on capital in each country is now a function of the amount of capital in that country, it will now matter who owns capital. In order for rates of return to be equal, then capital must be split evenly between the two countries.

The bond market gives country A a means of buying capital for use in country B - although the capital is nominally purchased by country B, its purchase is financed by borrowing from country A and its returns are used to pay off that debt.